3.10 Example: Geodesics on a sphere - paths in ϕ

we know that for a sphere the only non-zero christoffel symbols are $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$, and $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$.

Geodesic paths satisfy the equation

$$\frac{dx^A}{ds^2} + \Gamma^A_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = 0$$

suppose the path s is just a change in ϕ then $s = a\phi$ so there is no dependance on θ i.e. $x^1 = \theta$ =constant so $d\theta/ds = 0$ and $d^2\theta/ds^2 = 0$. While for ϕ we have $d\phi/ds = 1/a$ and $d^2\phi/ds^2 = 0$.

the LHS of the geodesic equation in ϕ is

$$\frac{d^2\phi}{ds^2} + \Gamma^{\phi}_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = 0 + \Gamma^{\phi}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} + \Gamma^{\phi}_{\phi\theta}\frac{d\phi}{ds}\frac{d\theta}{ds} = 0$$

so this looks good but we have to do BOTH coordinates.

the LHS of the geodesic equation in θ then

$$\frac{d^2\theta}{ds^2} + \Gamma^{\theta}_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = \frac{d^2\theta}{ds^2} + \Gamma^{\theta}_{\phi\phi}\frac{d\phi}{ds}\frac{d\phi}{ds} = 0 - \sin\theta_0\cos\theta_0 \times 1/a^2$$

this is only equal to 0 i.e. is only a geodesic for the special case of $\sin\theta_0\cos\theta_0 = 0$ i.e. $\theta_0 = 0, \pi/2, \pi$

$0, \pi$ are the north and south poles respectively, where a path in ϕ is just a point. the only geodesics which involve any distance is the equator - a great circle like before.

3.11 Easier Christoffel Symbols and geodesic pathss

calculating the Christoffel symbols was utterly tedious. even for the simplest 2D example there was a lot of 'turn the handle' maths that sooner or later will go wrong and we'll drop a term through loss of concentration from being so bored. And finding geodesics was worse! we were picking parameter curves to explore as the full generality was so tricky....

An easier way for both (note I didn't say EASY) is to use the Euler–Lagrange equations. We could have solved for the geodesics (at least for matter particles) by saying that these are the paths which give the shortest distance between two points, i.e. we are looking for the extremal path which has

$$\delta \int ds = 0$$

But this is often what we do in classical mechanics, where we look for the minimum energy path by getting the Lagrangian $L = T - V$ as the sum of kinetic T and potential V energies, and then finding the minimum energy path by integrating this over time i.e. $\delta \int L dt = 0$. For geodesics then there are no potential energy terms so $V = 0$ so we are looking only at kinetic energy, so $\delta \int T dt = 0$.

We'll do it in terms of per unit mass so $T = 1/2[(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2]$, or in metric terms (Euclidean 3D non-relativistic flat space)

$$T = \frac{1}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j = \frac{1}{2} \Big(\frac{ds}{dt}\Big)^2$$

So the minimum energy condition (Hamiltons principle in classical mechanics) gives

$$\delta \int T dt = \delta \int \Big(\frac{ds}{dt}\Big)^2 dt = 0$$

i.e. this is basically the same as the minimum path requirement which defines our geodesic which is $\delta \int (ds/dt) dt = 0$.

We know that in classical mechamics that the solution with the minimum energy satisfies the Euler-Lagrange equations i.e.

$$\frac{d}{dt} \Big(\frac{\partial L}{\partial \dot{x}^i}\Big) - \frac{\partial L}{\partial x^i} = 0$$

where $L = 1/2 g_{ij} \dot{x}^i \dot{x}^j$. If we wanted we could do long and tedious algebra on these equations to get the equation for a geodesic path, and it would turn out to be of the same form as we got before with far less work by doing parallel transport of a vector i.e.

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$

or, writing it explicitally

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

While there are obviously some technical problems in carrying this over into our 4D spacetime, where the metric is indefinite (can be +ve, 0 or -ve), if you do it, then it comes out to be the same as above except we are now going to take our derivative wrt some affine parameter u linearly related to the invariant path length, s, rather than to the COORDINATE t. So we could write the geodesic equation as before but where the dot now stands for derivative wrt the affine parameter u. So the geodesic paths are $\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$.

But then the midstep Euler–Lagrange equations still hold, so we have the condition that

$$\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{x}^\alpha}\Big) - \frac{\partial L}{\partial x^\alpha} = 0$$

And these equations give us directly the Christoffel symbols by comparison with the geodesic equations.

3.12 Example on sphere!

geodesic equations are

$$\frac{dx^A}{ds^2} + \Gamma^A_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = 0$$

but the equivalent Euler-Lagrange equations are

$$\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{x}^\alpha}\Big) - \frac{\partial L}{\partial x^\alpha} = 0$$

The E-L equations DON'T involve Christoffel symbols but the geodesic equations do. Yet both purport to give geodesic paths so both must ultimately be the same. so for the sphere $ds^2 = g_{AB}dx^A dx^B = a^2 d\theta^2 + a^2\sin^2\theta d\phi^2$ so

$$L = \frac{1}{2}g_{AB}(dx^A/ds)(dx^B/ds) = \frac{1}{2}g_{AB}\dot{x}^A\dot{x}^B = \frac{1}{2}(a^2\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2)$$

Do the E-L equation for θ:

$$\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{\theta}}\Big) - \frac{\partial L}{\partial \theta} = 0$$

do each bit separately so $\partial L/\partial\dot\theta = \frac{1}{2}a^2 2\dot\theta = a^2\dot\theta$ and $\partial L/\partial\theta = \frac{1}{2}a^2\dot\phi^2\partial\sin^2\theta/\partial\theta = a^2\dot\phi^2\sin\theta\cos\theta$ put these into the E-L equation and get

$$\frac{d(a^2\dot\theta)}{ds} - a^2\dot\phi^2\sin\theta\cos\theta = 0$$

$$\ddot\theta - \sin\theta\cos\theta\dot\phi^2 = 0$$

now we compare this with the equivalent geodesic equation

$$\ddot\theta + \Gamma^\theta{}_{BC}\dot x^B\dot x^C = 0$$

expand out the double sum

$$\ddot\theta + \Gamma^\theta{}_{\theta\theta}\dot\theta\dot\theta + \Gamma^\theta{}_{\phi\theta}\dot\phi\dot\theta + \Gamma^\theta{}_{\theta\phi}\dot\theta\dot\phi + \Gamma^\theta{}_{\phi\phi}\dot\phi\dot\phi = 0$$

Equate coefficients and see that $\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$ and all the rest of the $\Gamma^\theta{}_{BC} = 0$!

3.13 equivalence of geodesic and E-L equations

The Euler-Lagrange equations are

$$\frac{d}{du}\Big(\frac{\partial L}{\partial\dot x^c}\Big) - \frac{\partial L}{\partial x^c} = 0$$

where dot denotes derivative wrt some affine parameter u, $L = \frac{1}{2}g_{ab}\dot x^a\dot x^b$ and g_{ab} is a metric which depends only on position (i.e. x^c) and not velocity (i.e. $\dot x^c$ where x^c denotes any coordinate. do the two bits separately

$$\frac{\partial L}{\partial\dot x^c} = \frac{\partial(\frac{1}{2}g_{ab}\dot x^a\dot x^b)}{\partial\dot x^c} = \frac{1}{2}\dot x^a\dot x^b\frac{\partial g_{ab}}{\partial\dot x^c} + \frac{1}{2}g_{ab}\dot x^a\frac{\partial\dot x^b}{\partial\dot x^c} + \frac{1}{2}g_{ab}\dot x^b\frac{\partial\dot x^a}{\partial\dot x^c}$$

$$= 0 + \frac{1}{2}g_{ab}\dot x^a\delta^b_c + \frac{1}{2}g_{ab}\dot x^b\delta^a_c = \frac{1}{2}g_{ac}\dot x^a + \frac{1}{2}g_{cb}\dot x^b = g_{cb}\dot x^b$$

$\partial_c L = \partial_c(\frac{1}{2}g_{ab}\dot x^a\dot x^b) = \frac{1}{2}\dot x^a\dot x^b\partial_c g_{ab}$.

put this into the E-L equations and they become

$$\frac{d}{du}(g_{cb}\dot x^b) - \frac{1}{2}\dot x^a\dot x^b\frac{\partial g_{ab}}{\partial x^c} = 0$$

$$g_{cb}\ddot{x}^b + \dot{x}^b\frac{dg_{cb}}{du} - \frac{1}{2}\dot{x}^a\dot{x}^b\partial_c g_{ab} = 0$$

$$g_{cb}\ddot{x}^b + \dot{x}^b\dot{x}^a\partial_a g_{cb} - \frac{1}{2}\dot{x}^a\dot{x}^b\partial_c g_{ab} = 0$$

$$g_{cb}\ddot{x}^b + \frac{1}{2}\dot{x}^b\dot{x}^a\partial_a g_{cb} + \frac{1}{2}\dot{x}^b\dot{x}^a\partial_a g_{cb} - \frac{1}{2}\dot{x}^a\dot{x}^b\partial_c g_{ab} = 0$$

but $\dot{x}^b\dot{x}^a\partial_a g_{cb} = \dot{x}^d\dot{x}^a\partial_a g_{cd}$ and $\dot{x}^b\dot{x}^a\partial_a g_{cb} = \dot{x}^a\dot{x}^d\partial_d g_{ca}$ and $\dot{x}^a\dot{x}^b\partial_c g_{ab} = \dot{x}^a\dot{x}^d\partial_c g_{ad}$ as these are dummy indices so we can relabel them to whatever we want and now we can get everything in terms of $\dot{x}^a\dot{x}^d$ so

$$g_{cb}\ddot{x}^b + \frac{1}{2}\dot{x}^d\dot{x}^a(\partial_a g_{cd} + \partial_d g_{ca} - \partial_c g_{ad}) = 0$$

multiply by the contravariant metric g^{fc}

$$g^{fc}g_{cb}\ddot{x}^b + \frac{1}{2}g^{fc}(\partial_a g_{cd} + \partial_d g_{ca} - \partial_c g_{ad})\dot{x}^d\dot{x}^a = 0$$

$$\delta^f_b\ddot{x}^b + \Gamma^f{}_{ad}\dot{x}^d\dot{x}^a = 0$$

$$\ddot{x}^f + \Gamma^f{}_{ad}\dot{x}^d\dot{x}^a = 0$$

3.14 Geodesic on a sphere: EL in ϕ

$$ds^2 = a^2d\theta^2 + a^2\sin^2\theta d\phi^2$$

$$L = \frac{1}{2}(a^2\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2)$$

E-L equation in ϕ

$$\frac{d}{ds}\Big(\frac{\partial L}{\partial\dot{\phi}}\Big) - \frac{\partial L}{\partial\phi} = 0$$

$$\frac{\partial L}{\partial\dot{\phi}} = a^2\sin^2\theta\dot{\phi}$$

$$\frac{\partial L}{\partial\phi} = 0$$

$$\frac{d}{ds}(a^2 \sin^2\theta\dot{\phi}) - 0 = 0$$

$$\frac{d}{ds}(\sin^2\theta\dot{\phi}) = 0$$

$$\frac{d\sin^2\theta}{ds}\dot{\phi} + \sin^2\theta\frac{d\dot{\phi}}{ds} = 0$$

$$\frac{\partial(\sin^2\theta)}{\partial\theta}\frac{d\theta}{ds}\dot{\phi} + \sin^2\theta\ddot{\phi} = 0$$

$$2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + \sin^2\theta\ddot{\phi} = 0$$

$$\ddot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0$$

so this is how we find geodesic paths using the EL equations.
compare this with the geodesic equation in ϕ

$$\ddot{\phi} + \Gamma^{\phi}{}_{BC}\dot{x}^B\dot{x}^C = 0$$

$$\ddot{\phi} + \Gamma^{\phi}{}_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^{\phi}{}_{\theta\phi}\dot{\theta}\dot{\phi} + \Gamma^{\phi}{}_{\phi\theta}\dot{\phi}\dot{\theta} + \Gamma^{\phi}{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0$$

equate coefficiants and read off to see that

$$\Gamma^{\phi}{}_{\theta\phi}\dot{\theta}\dot{\phi} + \Gamma^{\phi}{}_{\phi\theta}\dot{\phi}\dot{\theta} = 2\cot\theta\dot{\phi}\dot{\theta}$$

and since $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$ then $\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta$