4 Tensor and Physical Curvature

So we've done a lot about tensor derivatives, and how this enables us to find the geodesic paths (= inertial frames, where we can do physics). But we also needed a way to describe curvature and we are STILL not able to do that. We've seen the metric tensor g_{ab} and the Christoffel symbols $\Gamma^a{}_{bc}$ both contain all the important information about the space curvature, but that they both also contain the unimportant information about what coordinates we are working in. We still need a nice way to quantify curvature so we can write gravity=curvature. There is one (but it isn't nice!).

4.1 Riemann Curvature Tensor

Key thing about curvature is that neighbouring geodesics get further apart (or closer together) at a rate depending on the local curvature. Flat space, no forces, geodesics are straight lines. The distance between two geodesics then increases/decreases linearly as a function of path length along the geodesic. So the second derivative of the distance between them is zero. For curved space then the geodesics don't separate linearly - the second derivative will not be zero, and this gives us a way to quantify the curvature of spacetime.

Take two geodesics γ and $\tilde{\gamma}$ with coodinates $x^a(u)$ and $\tilde{x}^a(u)$. The distance between them is $\zeta^a(u) = \tilde{x}^a(u) - x^a(u) =$. These are geodesics so

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

$$\frac{d^2\tilde{x}^a}{du^2} + \tilde{\Gamma}^a_{bc}\frac{d\tilde{x}^b}{du}\frac{d\tilde{x}^c}{du} = 0$$

but

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \frac{\partial \Gamma^a_{bc}}{\partial x^d}(\tilde{x}^d - x^d) = \Gamma^a_{bc} + \partial_d \Gamma^a_{bc}\zeta^d$$

so subtracting the geodesics gives

$$\frac{d^2\tilde{x}^a}{du^2} + \tilde{\Gamma}^a_{bc}\frac{d\tilde{x}^b}{du}\frac{d\tilde{x}^c}{du} - \frac{d^2x^a}{du^2} - \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

$$\frac{d^2(x^a+\zeta^a)}{du^2} + (\Gamma^a_{bc} + \partial_d\Gamma^a_{bc}\zeta^d)\frac{d(x^b+\zeta^b)}{du}\frac{d(x^c+\zeta^c)}{du} - \frac{d^2x^a}{du^2} - \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

Let dot denote derivative with respect to path length, and keep only first order terms i.e. ignore anything with two ζ terms.

$$\ddot{\zeta}^a + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

We have this term $\ddot{\zeta}^a$. We are looking for a tensor equation, and we'd get something with this term that transformed as a tensor if we had $D^2\zeta^a/du^2$.

$$\frac{D^2\zeta^a}{du^2} = \frac{D}{du}(\dot{\zeta}^a + \Gamma^a_{bc}\zeta^b\dot{x}^c) = \frac{D}{du}\dot{\zeta}^a + \frac{D}{du}(\Gamma^a_{bc}\zeta^b\dot{x}^c)$$

$$= \ddot{\zeta}^a + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \frac{d}{du}(\Gamma^a_{bc}\zeta^b\dot{x}^c) + \Gamma^a_{ef}(\Gamma^e_{bc}\zeta^b\dot{x}^c)\dot{x}^f$$

So we can substitute our expression for $D^2\zeta^a/du$ (which is a tensor) for $d^2\zeta^a/du$ (which is NOT a tensor). and after lots of tedious algebra and index relabling we get:

$$\frac{D^2\zeta^a}{du^2} + (\Gamma^a_{be}\Gamma^e{}_{cd} - \partial_d\Gamma^a{}_{bc} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} + \partial_b\Gamma^a{}_{dc})\zeta^b\dot{x}^c\dot{x}^d = 0$$

So here we have something which should equal zero if the space is flat, and not zero if the space is curved. This can all be stuck together into the Riemann curvature tensor

$$\frac{D^2\zeta^a}{du^2} + R^a_{cbd}\zeta^b\dot{x}^c\dot{x}^d = 0$$

where

$$R^a_{cbd} = (\Gamma^a_{be}\Gamma^e_{cd} - \partial_d\Gamma^a_{bc} - \Gamma^a_{ed}\Gamma^e_{bc} + \partial_b\Gamma^a_{dc})$$

This tensor DOES NOT CARE about coordinate systems. Flat space has all components of $R^a{}_{bcd} = 0$ irrespective of whether you are working in spherical polar coordinates or cartesian coordinates. And if all components are zero then $D^2\zeta^a/du^2 = 0$ and $\zeta^a = Au + B$ so geodesics in flat space separate linearly (or remain parallel if A=0). Geodesics in curved space do not.

4.2 Symmetry properties of the Riemann curvature tensor

We could write this out in full but its very long and tedious so instead we are going to use a trick. We can always define locally geodesic coordinates, those in which the metric tensor is given by $\eta_{\alpha\beta}$ and the Christoffel symbols are zero, BUT THEIR DERIVATIVES ARE NOT - we can't transform gravity away in a global sense. I'd call this the primed frame but the equations will get very messy. BUT IN WHAT FOLLOWS WE ARE LOOKING AT THE COMPONENTS OF THE CURVATURE TENSOR IN THE LOCAL GEODESIC FRAME.

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb}$$

Start with the fully covariant quantity

$$R_{abcd} = g_{ae} R^e{}_{bcd}$$

$$R_{abcd} = g_{ae} [\partial_c \Gamma^e{}_{db} - \partial_d \Gamma^e{}_{cb}]$$

we can take the g_{ae} inside the partial derivatives as in these local geodesic coordinates then covariant derivative reduces to partial derivative!! this gives

$$R_{abcd} = \partial_c \Gamma_{adb} - \partial_d \Gamma_{acb}$$

these Christoffel symbols can be written in terms of the derivatives of the metric (see lecture 7) where we had

$$2\Gamma_{abc} = \partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc}$$

so

$$2R_{abcd} = \partial_c [\partial_d g_{ab} + \partial_b g_{da} - \partial_a g_{db}] - \partial_d [\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{cb}]$$

$$R_{abcd} = \frac{1}{2} (\partial_c \partial_b g_{da} - \partial_c \partial_a g_{db} - \partial_d \partial_b g_{ca} + \partial_d \partial_a g_{cb})$$

where we've used the fact that $g_{ab} = g_{ba}$ and that $\partial_d \partial_c g_{ab} = \partial_c \partial_d g_{ab}$.

So then we can see that

$$R_{abcd} = -R_{bacd}$$

$$R_{abcd} = -R_{abdc}$$

$$R_{abcd} = R_{cdab}$$

These are TENSOR equations, so must be true in all frames.

But this then shows us that there are a lot of $R_{abcd} = 0$ as if we have a repeated index in the first 2 places then $R_{aacd} = -R_{aacd}$ and the only way that a number can be equal to its own negative if if its zero. Similarly for a repeated index on the last two places $R_{abcc} = -R_{abcc} = 0$.

Cyclic Identity

So you can see that so if we cyclically perturb the covariant indices then we have

$$R^a{}_{cdb} = \partial_d \Gamma^a{}_{bc} - \partial_b \Gamma^a{}_{dc}$$
$$R^a{}_{dbc} = \partial_b \Gamma^a{}_{cd} - \partial_c \Gamma^a{}_{bd}$$

so add all these, remembering that christoffel symbols are symmetric

$$R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \partial_d \Gamma^a{}_{bc} - \partial_b \Gamma^a{}_{dc} + \partial_b \Gamma^a{}_{cd} - \partial_c \Gamma^a{}_{bd} = 0$$

But $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$ is a TENSOR equation. So even though we derived it in the local frame, it must be true in ALL frames!