PHYS2581 Foundations2A: QM 9 solution

(a) $J^2 = (\underline{L} + \underline{S}).(\underline{L} + \underline{S}) = L^2 + S^2 + 2\underline{L}.\underline{S}$ (S and L commute so order does not matter)

hence $\underline{L}.\underline{S} = \frac{1}{2}(J^2 - L^2 - S^2)$ [S: 1 mark]

(b) $L_+S_- = (L_x + iL_y)(S_x - iS_y) = L_xS_x - iL_xS_y + iL_yS_x + L_yS_y$

$L_-S_+ = (L_x - iL_y)(S_x + iS_y) = L_xS_x + iL_xS_y - iL_yS_x + L_yS_y$ [U: 1 mark]

$$\frac{1}{2}(L_+S_- + L_-S_+) + L_zS_z = \frac{1}{2}(2L_xS_x + 2L_yS_y) + L_zS_z = L_xS_x + L_yS_y + L_zS_z$$

as required [U: 1 mark]

(c) $\underline{L}.\underline{S}\psi^0_{2,1,-1,1/2} = [\frac{1}{2}(L_+S_- + L_-S_+) + L_zS_z]R_{21}Y_{1,-1}X_+$

$= R_{21}[\frac{1}{2}(L_+S_- + L_-S_+) + L_zS_z]Y_{1,-1}X_+$ as none of these operators affect n, l (they commute) [U: 1 mark]

take each operator in turn - all S commute with all L so order doesn't matter. all S operators only affect X while L operators only affect Y

$L_zS_zY_{1,-1}X_+ = (L_zY_{1,-1})(S_zX_+) = (L_zY_{1,-1})\frac{\hbar}{2}X_+ = \frac{\hbar}{2}.(-\hbar)Y_{1,-1}X_+$

$= -\frac{\hbar^2}{2}Y_{1,-1}X_+$ [U: 1 mark]

$L_-S_+Y_{1,-1}X_+ = (L_-Y_{1,-1})(S_+X_+) = 0$ as can't raise m_s above 1/2 (and can't lower m below -1 for l=1)

$L_+S_-Y_{1,-1}X_+ = (L_+Y_{1,-1})(S_-X_+) = (L_+Y_{1,-1})\frac{\hbar}{\sqrt{2}}X_-$

$A_{1,-1} = \hbar\sqrt{2 - 1(-1+1)} = \hbar\sqrt{2}$ [U: 1 mark]

so $\frac{1}{2}(L_+S_- + L_-S_+)Y_{1-1}X_+$ is

$\frac{1}{2}(L_+Y_{1,-1})(\frac{\hbar}{\sqrt{2}}X_-) = \frac{1}{2}\hbar\sqrt{2}\frac{\hbar}{\sqrt{2}}Y_{1,0}X_- = \frac{\hbar^2}{2}Y_{1,0}X_-$

total $= R_{21}\frac{\hbar^2}{2}(Y_{1,0}X_- - Y_{1,-1}X_+) = \frac{\hbar^2}{2}(\psi^0_{2,1,0,-1/2} - \psi^0_{2,1,-1,1/2})$

the operator does not return the same function that we gave it, hence the unperturbed energy eigenfunctions are not eigenfunctions of $\underline{L}.\underline{S}$ [U: 1 marks]

(d) $[L_xS_x + L_yS_y + L_zS_z, L_z] = [L_xS_x, L_z] + [L_yS_y, L_z] + [L_zS_z, L_z]$

$= S_x[L_x, L_z] + S_y[L_y, L_z] + S_z[L_z, L_z]$ as S, L commute [U: 1 mark]

$= S_x(-i\hbar L_y) + S_y i\hbar L_x = i\hbar(S_yL_x - S_xL_y) \neq 0$

$[L_xS_x + L_yS_y + L_zS_z, S_z] = [L_xS_x, S_z] + [L_yS_y, S_z] + [L_zS_z, S_z]$

$= L_x[S_x, S_z] + L_y[S_y, S_z] + L_z[S_z, S_z] = L_x(-i\hbar S_y) + L_y i\hbar S_x = i\hbar(L_yS_x - L_xS_y) \neq 0$ [U: 1 mark]

$[L_xS_x + L_yS_y + L_zS_z, J_z] = [L_xS_x + L_yS_y + L_zS_z, L_z] + [L_xS_x + L_yS_y + L_zS_z, S_z]$ so $i\hbar(S_yL_x - S_xL_y + L_yS_x - L_xS_y) = 0$

So J_z commutes with the perturbation whereas L_z and S_z do not. so we should use n, l, j, m_j rather than n, l, m, m_s [S: 1 mark]