

ELEMENTS OF QUANTUM MECHANICS - EXAMPLES CLASS 4

The common eigenfunctions of the angular momentum operators L^2 and L_z are the spherical harmonics, Y_{lm} , such that $L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}$ and $L_z Y_{lm} = m\hbar Y_{lm}$.

The associated ladder operators are $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$, where $L_{\pm} Y_{lm} = (L_x \pm iL_y) Y_{lm} = A_{lm\pm} Y_{lm\pm 1}$ where $A_{lm\pm}$ is a constant which depends on l, m and which of L_+ or L_- we are using.

- (a) Use the definitions above to explicitly show that $[L^2, L_z] Y_{lm} = 0$. If $L^2 Y_{lm}$ is measured to be $30\hbar^2$, what are the possible values of a measurement of L_z ?
- (b) Show that $L_x = \frac{1}{2}(L_+ + L_-)$. Use this, together with the orthonormal properties of wavefunctions, to calculate $\langle L_x \rangle$ for any spherical harmonic Y_{lm} .
- (c) Give a general argument showing that the eigenfunctions of $L_+ L_-$ are $\propto Y_{lm}$. Rewrite $L_+ L_-$ in terms of $L^2 = L_x^2 + L_y^2 + L_z^2$ and L_z remembering $[L_x, L_y] = i\hbar L_z$. Hence write down the eigenvalues of $L_+ L_-$. Show explicitly that this is consistent with $A_{lm\pm} = \hbar\sqrt{l(l+1) - m(m\pm 1)}$
- (d) Use the definition of L_x in (b) above and the definition of $A_{lm\pm}$ in (c) above to find the values of the coefficients a, b, c in order that $L_x \psi = \hbar q \psi$ for $\psi = aY_{11} + bY_{10} + cY_{1-1}$. Solve this explicitly to find the normalised eigenfunctions of L_x for $q = 1, 0, -1$