Hydrogen has unperturbed energies $E_n^0 = E_1^0/n^2$. The ground state has unperturbed wavefunction $\psi_{100}^0 = (\pi a^3)^{-1/2} e^{-r/a}$. Hydrogen in the ground state is placed in an external electric field, E_{ext} (Stark effect), in the z direction, so there is an additional term to the potential energy of the electron of

$$H' = -eE_{ext}z = -eE_{ext}r\cos\theta$$

(a) Show that there is no first order correction to the ground state energy well potential, i.e. $E_1^1 = \int \psi_{100}^{0*} H' \psi_{100}^0 dV = 0$

(b) The second order correction to the ground state energy is given by

$$E_1^2 = \sum_{m\neq 1} \frac{|\int_V \psi_m^{0*} H' \psi_1^0 dV|^2}{E_1^0 - E_m^0}$$

where m represents a sum over all energy eigenstates. Calculate the contribution to the second order correction to the ground state energy from each of the $n = 2$ levels. [Hint: do the angle integrals first as these are all zero except for one! this saves you the trouble of doing the radial integral!]

$$\psi_{200} = \sqrt{\frac{1}{2\pi a}}\,\frac{1}{2a}(1 - \frac{r}{2a})e^{-r/2a}$$

$$\psi_{210} = \frac{1}{4\sqrt{2\pi a^3}}\,\frac{r}{a}e^{-r/2a}\cos\theta$$

$$\psi_{211} = -\frac{1}{8\sqrt{\pi a^3}}\,\frac{r}{a}e^{-r/2a}\sin\theta e^{i\phi} \qquad \psi_{21-1} = -\psi_{211}^*$$

(c) Calculate the contribution to the second order correction to the ground state energy from each of the $n = 3$ levels - the only one which does NOT have the angles integrate to zero is

$$\psi_{310} = \frac{2}{27}\sqrt{\frac{2}{\pi a^3}}\frac{r}{a}(1 - \frac{r}{6a})e^{-r/3a}\cos\theta$$

Hence estimate E_1^2 from the sum of the corrections from levels 2 and 3.

Useful integrals:

$$\int_0^\infty x^p e^{-qx} dx = \frac{p!}{q^{p+1}}$$

$$\int \cos\theta\sin\theta d\theta = \frac{-1}{2}\cos^2\theta \qquad \int \cos\theta\sin^2\theta d\theta = \frac{1}{3}\sin^3\theta \qquad \int \cos^2\theta\sin\theta d\theta = \frac{-1}{3}\cos^3\theta$$