PHYS2581 Foundations2A: QM3 answers

a) region I $V = 0$ so

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{-2mE}{\hbar^2}\psi = -k^2\psi$$

where $k^2 = 2mE/\hbar^2$

solution is $\psi(x) = A\sin kx + B\cos kx$ but boundary condition $\psi(0) = 0$ so $B = 0$

region II:

$$\frac{-\hbar^2}{2m}\frac{\partial\psi^2}{\partial x^2} = \frac{2m}{\hbar^2}(E - V_0)\psi$$

so

$$\frac{\partial\psi^2}{\partial x^2} = \rho^2\psi$$

where $\rho^2 = 2m(V_0 - E)/\hbar^2$

general solution $\psi = Ce^{\rho x} + De^{-\rho x}$ boundary condition $\psi(\infty) = 0$ is $C = 0$ so $\psi = De^{-\rho x}$.

[1 mark]

need to match wavefunction at $x = L$ so $A\sin kL = De^{-\rho L}$

need to match $d\psi/dx$ at $x = L$ so $Ak\cos kL = -\rho De^{-\rho L}$

divide: $k\cot kL = -\rho$ so $kL\cot kL = -\rho L$ [1 mark]

b) hence $z\cot z = -\rho L$ for $z = kL$

$\rho^2 L^2 = 2mL^2(V_0 - E)/\hbar^2 = 2mL^2V_0/\hbar^2 - 2mL^2E/\hbar^2 = z_0^2 - z^2$

so $z\cot z = -\sqrt{z_0^2 - z^2}$ or $\cot z = -\sqrt{(z_0/z)^2 - 1}$ as required [2 marks]

for $z_0 = 6$ then wolfram alpha gives $z_1 = 2.68$ and $z_2 = 5.23$ to 3 sig figs. hence there are 2 bound states [1 mark]

$E_n = z_n^2\hbar^2/(2mL^2)$ so these have energies $E_1 = 7.18$ and 27.3 in units of $\hbar^2/(2mL^2)$ [1 mark]

c) $\psi = A\cos kx$ for $0 < x < L$ and $De^{-\rho x}$ for $x > L$

$kL = z_1$ so $k = z_1/L = 2.68/L$ [1 mark]

$\rho^2 L^2 = z_0^2 - z_1^2$ so $\rho^2 L^2 = 36 - 7.18 = 28.82$ and $\rho = 5.37/L$ [1 mark]

$A\sin kL = De^{-\rho L}$ so $A\sin 2.68 = De^{-5.37}$ so $D = Ae^{5.37}\sin 2.68 = 95.7A$ [1 mark]

normalise - its piecewise continuous

$\int_0^L A^2 \sin^2 kxdx + A^2(95.7)^2 \int e^{-2\rho x}dx = 1$

$1/A^2 = \int_0^L \sin^2 kxdx + 9160 \int_L^\infty e^{-2\rho x}dx$

$= 0.574L + 9160e^{-537/50}\frac{50L}{537} = 0.574L + 0.0185L = 0.593L$

hence $A = 1.30/\sqrt{L}$ [1 mark]