ELEMENTS OF QUANTUM MECHANICS - EXAMPLES CLASS 2

$\Psi(x,t=0) = Ax(L-x)$ in the infinite square well potential for $0 < x < L$ and zero elsewhere, where $A = \sqrt{30/L^5}$. Eigenfunctions of this potential are $\psi_n(x) = N \sin n\pi x/L$, with $N = \sqrt{2/L}$ corresponding to energy $E_n = n^2\pi^2\hbar^2/(2mL^2)$.

i) $\Psi = \sum_n c_n\psi_n$ so $c_n = \int \psi_n^*\Psi dx$

$$c_n = AN\int_0^L Lx\sin n\pi x/L dx - AN\int_0^L x^2\sin n\pi x/L dx$$

$$c_n/(AN) = L[\frac{\sin(n\pi x/L) - n\pi x/L\cos(n\pi x/L)}{(n\pi/L)^2}]_0^L -$$

$$[\frac{(2-(n\pi/L)^2x^2\cos(n\pi x/L) + (2n\pi x/L)\sin(n\pi x/L)}{(n\pi/L)^3}]_0^L$$

$$= L^3\frac{-n\pi\cos(n\pi)}{(n\pi)^2} - L^3\Big(\frac{[2-(n\pi)^2]\cos(n\pi) - 2\cos(0)}{(n\pi)^3}\Big)$$

$$= \frac{L^3}{n^3\pi^3}\Big(-n^2\pi^2\cos(n\pi) - 2\cos(n\pi) + n^2\pi^2\cos(n\pi) + 2\cos(0)\Big)$$

$$= \frac{2L^3}{n^3\pi^3}(\cos(0) - \cos(n\pi))$$

$$c_n = \sqrt{2/L}\sqrt{30/L^5}\frac{2L^3}{n^3\pi^3}(\cos(0) - \cos(n\pi))$$

$$= \frac{4\sqrt{15}}{n^3\pi^3}(\cos(0) - \cos(n\pi))$$

ii) $c_1 = \frac{4\sqrt{15}}{\pi^3}(\cos(0) - \cos(\pi)) = 8\sqrt{15}/(\pi^3)$
$c_2 = \frac{4\sqrt{15}}{8\pi^3}(\cos(0) - \cos(2\pi)) = 0$
$c_3 = \frac{4\sqrt{15}}{27\pi^3}(\cos(0) - \cos(3\pi)) = 8\sqrt{15}/(27\pi^3)$
general form is $c_n = 0$ for even n, $c_n = 8\sqrt{15}/(n^3\pi^3)$ for odd

iii) prob state 1 is $c_1^2 = 64.15/\pi^6 = 0.999$

iv) $< E >= \sum_n c_n^2 E_n = \sum_{oddn} 64.15/(n^6\pi^6)n^2E_1 = \sum_n (64.15/\pi^6)E_1/n^4 = (64.15/96)E_1/\pi^2 = 10E_1/\pi^2$

bigger than E_1 becasue its not just in state 1, so the mean energy has to be bigger than that of state 1.

v) $< E >= \int -(\hbar^2/2m)A^2(Lx-x^2)d^2/dx^2(Lx-x^2)dx$
$= -(\hbar^2/2m)A^2\int(Lx-x^2)d/dx(L-2x)dx = -(\hbar^2/2m)A^2\int(Lx-x^2). -2dx$
$= 2(\hbar^2/2m)A^2\int(Lx-x^2)dx = 2(\hbar^2/2m)(30/L^5)\Big([Lx^2/2]_0^L - [x^3/3]_0^L\Big)$
$= 2(\hbar^2/2m)(30/L^5)(L^3/2 - L^3/3) = 2(\hbar^2/2m)(30/L^5)L^3(3-2)/6 = 2(\hbar^2/2m)(30/L^2)(1/6) = 10\hbar^2/(2mL^2) = 10/pi^2E_1$