PHYS2581 Foundations2A: QM prob4 solution

EQM2.4: i) $l = 3$ so $E_3 = 3(4)\hbar^2/(2I) = 6\hbar^2/I$ $-3 < m < 3$, degeneracy 7

$l = 4$ so $E_4 = 4(5)\hbar^2/(2I) = 10\hbar^2/I$ $-4 < m < 4$, degeneracy 9

$l = 5$ so $E_4 = 5(6)\hbar^2/(2I) = 15\hbar^2/I$ $-5 < m < 5$, degeneracy 11 [1 mark]

ii) probability of finding the electron in dV is

$$dP = Y^*_{2,-1}Y_{2,-1}\sin\theta d\theta d\phi = \left(\frac{15}{8\pi}\right)\sin^3\theta\cos^2\theta d\theta d\phi$$

$$\int_\phi dP = \int_{\phi=0}^{2\pi}\left(\frac{15}{8\pi}\right)\sin^3\theta\cos^2\theta d\theta d\phi = \left(\frac{15}{4}\right)\sin^3\theta\cos^2\theta d\theta$$

so probability density per unit θ is $\left(\frac{15}{4}\right)\sin^3\theta\cos^2\theta$ [1 mark]

max where derivative is zero. so

$$d/d\theta(\sin^3\theta\cos^2\theta) = 3\sin^2\theta\cos\theta\cos^2\theta + \sin^3\theta 2\cos\theta(-\sin\theta) = 0$$

$$3\sin^2\theta\cos^3\theta = 2\cos\theta\sin^4\theta$$

$$3\cos^2\theta = 2\sin^2\theta = 2(1-\cos^2\theta)$$

so $\cos\theta = \pm\sqrt{2/5}$ so $\theta = 50.8^\circ, 129^\circ$ [2 marks]

$$<\theta>= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{15}{8\pi}\right)\theta\sin^3\theta\cos^2\theta d\theta d\phi = \left(\frac{15}{4}\right)\frac{2\pi}{15} = \frac{\pi}{2}$$

[1 mark]

$$<\cos\theta>= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{15}{8\pi}\right)\sin^3\theta\cos^3\theta d\theta d\phi = 0$$

[1 mark]

prob electron in region $0 < \theta < \pi/3$ is

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/3} \left(\frac{15}{8\pi}\right) \sin^3\theta \cos^2\theta d\theta d\phi$$

$$= \frac{15}{4} 0.0979 = 0.367$$

[1 mark]

iii) eigenfunction if $L^2 Y_{lm} = cY_{lm}$ where c is a constant. $Y_{lm} = A\cos\theta\sin\theta e^{-i\phi}$ where $A = (15/8\pi)^{1/2}$

$$\mathbf{L}^2 Y_{lm} = -A\hbar^2 \left\{ \frac{e^{-i\phi}}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial(\sin\theta\cos\theta)}{\partial\theta} \right) + \frac{\sin\theta\cos\theta}{\sin^2\theta} \frac{\partial^2(e^{-i\phi})}{\partial\phi^2} \right\}$$

$$= -\hbar^2 A \left\{ \frac{e^{-i\phi}}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta(\cos^2(\theta) - \sin^2\theta) \right) + \frac{\sin\theta\cos\theta}{\sin^2\theta} (-i)^2 e^{-i\phi} \right\}$$

$$= -\hbar^2 A e^{-i\phi} \left\{ \frac{1}{\sin\theta} (\cos^3\theta - 5\sin^2\theta\cos\theta) - \frac{\cos\theta}{\sin\theta} \right\}$$

$$= -\hbar^2 A e^{-i\phi} \left\{ \frac{1}{\sin\theta} (\cos\theta(1-\sin^2\theta) - 5\sin^2\theta\cos\theta - \cos\theta) \right\}$$

$$= -\hbar^2 A e^{-i\phi}(-6\sin\theta\cos\theta) = 6\hbar^2 Y_{2,-1}$$

so this is an eigenfunction of L^2 [2 marks]

$$L_z Y_{2,-1} = -i\hbar \frac{\partial}{\partial\phi} A \sin\theta\cos\theta e^{-i\phi} = -iA\sin\theta\cos\theta\hbar(-i)e^{-i\phi} = -\hbar Y_{2,-1}$$

so this is an eigenfunction of L_z

$$L_x Y_{2,-1} = i\hbar A \left(e^{-i\phi} \sin\phi \frac{\partial(\sin\theta\cos\theta)}{\partial\theta} + \cot\theta\cos\phi(\sin\theta\cos\theta) \frac{\partial e^{-i\phi}}{\partial\phi} \right)$$

$$= i\hbar A \left(e^{-i\phi} \sin\phi(\cos^2\theta - \sin^2\theta) - i\cos^2\theta\cos\phi e^{-i\phi} \right)$$

$$= i\hbar A e^{-i\phi} \left(\sin\phi(\cos^2\theta - \sin^2\theta) - i\cos^2\theta\cos\phi \right)$$

this isn't going to go to $Y_{2,-1}$ so this is not an eigenfunction of L_x [1 mark]