1.9 Lorentz transformations

We have an event at point P which is x, y, z, t in frame S and x', y', z', t' in frame S'. who do we relate these co-ordinates to each other if the origins $O - O'$ are co-incident at $t = t' = 0$ where S' moves relative to S in the +ve x direction at speed u.

Lets make it more clear. The distance O' to P is $x' = L'$ is proper distance in S' This will be seen in S as length contracted to $L = L'/\gamma = x'/\gamma$

hence the distance O to P as measured in S is the speed of the frame plus the length contracted distance measured in S i.e.

$x = ut + L = ut + x'/\gamma$

solve for x' and get

$x' = \gamma(x - ut)$

and then we can just write down what happens for S' to S (swap u for -u and primes for unprimes but REMEMBER time

so start at the same place as before but swap to get $x' = -ut' + x/\gamma$

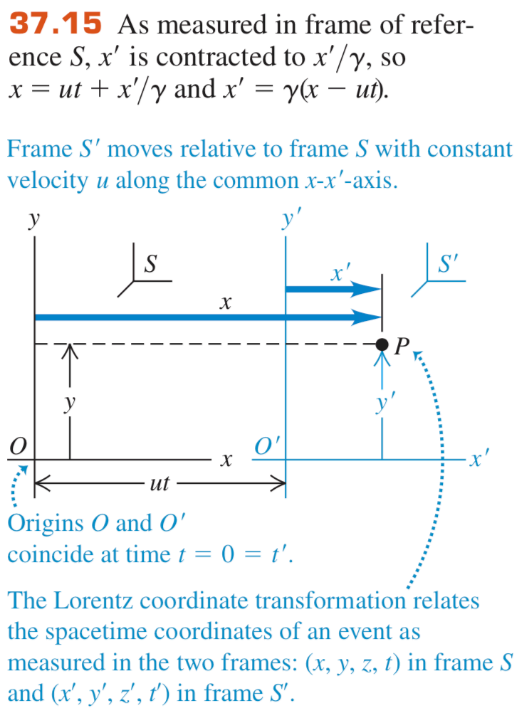

Figure 6:

so we have two equations for x' so we can eliminate and solve to get how t relates to t'

$$-ut' + x/\gamma = \gamma x - \gamma ut$$

$$-\gamma ut' + x = \gamma^2 x - \gamma^2 ut$$

$$\gamma ut' = x(1 - \gamma^2) + \gamma^2 ut$$

but we have $1 - \gamma^2 = 1 - 1/(1 - \beta^2) = (1 - \beta^2 - 1)/(1 - \beta^2) = -\beta^2\gamma^2$ so

$$\gamma ut' = \gamma^2 ut - \beta^2\gamma^2 x$$

$$t' = \gamma(t - xu/c^2)$$

bring everything together and we have the Lorentz transformations

$$x' = \gamma(x - ut) \quad y' = y \quad z' = z \quad t' = \gamma(t - xu/c^2)$$

or we can write these for quantities in S given those in S' as (replace $-u$ or u and swap primed/unprimed)

$$x = \gamma(x' + ut') \quad y = y' \quad z = z' \quad t = \gamma(t' + x'u/c^2)$$

For values of $u \ll c$ then $\gamma(u) \to 1$ and we get back to the galilean transformations of classical mechanics. but for $u \to c$ then space and time become intertwined and we can no longer say that they are absolutes independent of the frame of reference. The only SPECIAL frame of reference is the rest frame.

the combination x, y, z, t is called the spacetime coordinates of an event as measured in S. They relate to spacetime coordinates in frame S' via these transformations.

Example YF37.6. Mavis pilots her spacecraft across a finish line at 0.6c relative to the line, winning the race. (event 1). According to her, at the same instant as she, at the front of the ship, sees herself crossing the line, a 'hooray' message is sent from the back of the ship (event 2). She measures the length of her ship to be 300m. Stanley is at the finish line, and is at rest relative to it. When and where does he measure events 1 and 2

Set up axes in the standard way. Mavis is in S' as here she is at rest relative to the ship. let the origins coincide at $t = t' = 0$ when Mavis crosses the line. Then event 1 in S' is at $x'_1, y'_1, z'_1, t'_1 = 0, 0, 0, 0$ and event 2 in S' is at $-300, 0, 0, 0$ as the events are simultaneous in this frame and the ships length is 300m in this frame.

First task is always calculate γ for the problem. $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{(}1 - 0.6^2) = 1.25$

In S event 1 is at $x_1 = \gamma(x'_1 + ut_1) = 0$ and $t_1 = t'_1 = 0$

In S, event 2 is as $x_2 = \gamma(x'_2 + ut_2) = 1.25 \times -300 = -375$ m and is seen at time $t_2 = \gamma(t'_2 + x'_2 u/c^2) = 1.25 \times (0 - 300 \times 0.6/c = -225/c = -7.5 \times 10^{-7}\ s$

And here its negative, so in S, the hooray message comes 0.75μs BEFORE SHE HAS WON!

But its not violating causality because the signal from the back of the spacecraft was not triggered by Mavis. It was an unconnected event.

If instead, Mavis sends a signal to the back of the spacecraft to trigger the 'hurray' then it adds an extra light travel time of $300/c = 1 \times 10^{-6}$s to get from the front to the back of the spacecraft. Hence event 2 in her frame, S', has coordinates $-300, 0, 0, 10^{-6}$

Stanley then measures this new (causal) event 2 happening at time $t_2 = \gamma(t'_2 + x'_2 u/c^2) = 1.25 \times (10^{-6} - 300 \times 0.6/c) = 5 \times 10^{-7}$ s

This is now positive, so causality is actually preserved!

Causality is what we care about because its about a physical connection. Simulteneity is just a question of the observers point of view - observers can disagree about whether an object behind me is seen directly in line with me or projected to my left or my right. its not an important difference, everyones view is 'correct'. but if I am HOLDING the object (so I can affect it physically) then all observers agree that its in line with me.

1.10 Lorentz velocity transformations

We had the Lorentz transformations between coordinates

$$x' = \gamma(x - ut) \quad y' = y \quad z' = z \quad t' = \gamma(t - xu/c^2)$$

but we can re-write them considering only a small displacement dx, dy, dz, dt and get

$$dx' = \gamma(dx - udt) \quad dy' = dy \quad dz' = dz \quad dt' = \gamma(dt - dxu/c^2)$$

$$\begin{aligned} dx'/dt' &= \gamma(dx - udt)/\gamma(dt - dxu/c^2) \\ &= \frac{dx/dt - u}{1 - dx/dtu/c^2} \\ v'_x &= \frac{v_x - u}{1 - v_x u/c^2} \end{aligned}$$

This has some interesting properties! when $u, v_x \ll c$ then $v'_x \to v_x - u$ as expected. but when we go to $v_x \to c$ then $v'_x \to c$ also - lets see this explicitly by setting $v_x = c$

$$v'_x = \frac{c - u}{1 - cu/c^2} = c\frac{c - u}{c - u} = c$$

Anything moving with velocity $\to c$ in S also has velocity $v'_x \to c$ in S' despite the relative notion of the two frames. The speed of light is the same in any frame (by construction)

We get the inverse transforms as ever by swapping primes and unprimes and $-u$ for u.

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2}$$