37.16 The spaceship, robot space probe, and scoutship.

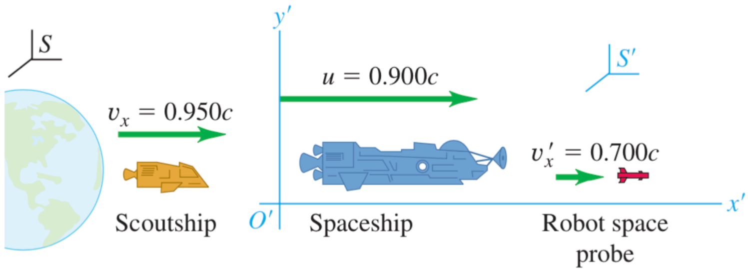

Figure 7:

Example YF37.7 spacecraft moving away from earth at 0.9c fires robot space probe in same direction as its motion at 0.7c as seen from the spacecraft. What is the velocity as seen from the Earth?

Set up the frames. The spacecraft is S', and its velocity relative to earth, S, is $u = 0.9c$.

The robot probe has $v'_x = 0.7c$ - its relative to the spacecraft so its in S'

This would be seen in S as $v_x = \frac{0.7c+0.9c}{1+0.7\times0.9} = 0.982c$

A scoutship is sent from Earth at 0.95c to try to catch up with the spacecraft. What is the speed of the spacecraft with respect to the scout?

We know the scout ship in the Earth frame S has velocity $0.95c$ while the spacecraft has $0.9c$. So the scout is catching up with the spacecraft. We want to measure the scout speed from the spacecraft so the spacecraft should be the one at rest, so $u = 0.9c$

Their relative velocity in frame S' is

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} = c\frac{0.95 - 0.9}{1 - 0.9 \times 0.95} = 0.345c$$

What is the speed of the probe relative to the scout?

We know that the probe is moving at velocity $0.982c$ in the Earth frame. While the scout has $0.95c$ in this frame so the scout will be falling behind. So make S' be the probe rest frame so $u = 0.982c$

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} = c\frac{0.95 - 0.982}{1 - 0.95 \times 0.982} = -0.477c$$

1.11 velocity transformations in orthoganal directions

While the coordinates transformation $y - y'$ and $z - z'$ are not affected by a velocity boost on the x-axis, this is not true for $v_y - v'_y$ and $v_z - v'_z$. This is because time dilation affects motion along all axes.

$$x' = \gamma(x - ut) \quad y' = y \quad z' = z \quad t' = \gamma(t - xu/c^2)$$

but we can re-write them considering only a small displacement dx, dy, dz, dt and get

$$dx' = \gamma(dx - udt) \quad dy' = dy \quad dz' = dz \quad dt' = \gamma(dt - dxu/c^2)$$

$$v'_y = dy'/dt' = \frac{dy}{\gamma(dt - dxu/c^2)} = \frac{dy/dt}{\gamma(1 - dx/dtu/c^2)} = \frac{v_y}{\gamma(1 - v_x u/c^2)}$$

$v'_y \neq v_y$! because even though $dy' = dy$, $dt' \neq dt$ - its the time change between frames which leads to a velocity change!

1.12 Doppler Effect

We could do this by considering lengths and times as in the book. Or we could just do the transformations! Suppose we have an electromagnetic wave. $E'(x', t') = A\cos(k'x' - \omega't')$ and we know that this travels at $c = \omega'/k' = f'\lambda'$ now do the transformation to frame S:

$E(x,t) \propto \cos[k'\gamma(x-ut) - \omega'\gamma(t-ux/c^2)]$

$= \cos[\gamma(k' + \omega' u/c^2)x - \gamma(\omega' + uk')t]$

but $c = \omega'/k'$ so $ck' = \omega'$ and we have

$= \cos[\gamma(k' + k'u/c)x - \gamma(\omega' + u\omega'/c)t]$

$= \cos[\gamma(1+u/c)k'x - \gamma(1+u/c)\omega' t] = \cos(kx - \omega t)$ where we have k and ω in the S frame. these relate to k' and ω' via the factor

$$\gamma(1+u/c) = \frac{1+u/c}{\sqrt{1-u^2/c^2}}$$

$$= \frac{1+u/c}{\sqrt{(1-u/c)(1+u/c)}} = \sqrt{\frac{1-u/c}{1+u/c}}$$

so we have

$$\omega = \sqrt{\frac{1-u/c}{1+u/c}}\omega' \qquad k = k'\sqrt{\frac{1-u/c}{1+u/c}}$$

and $c = \omega'/k' = \omega/k = (2\pi f)(\lambda/2\pi) = f\lambda$. The wavelength gets longer (frequency gets lower) when its moving away (+u) and shorter (higher frequency) when its moving towards you (-u). But its NOT just a simple γ factor as lengths contract, changing the wavelength as well as time dilating, changing the frequency.

2 Relativisitic kinematics

This all has implications for how we do kinematics, how we define momentum, energy, force - just about everything changes now as $u \to c$.

One of the key assumptions (postulates) was that physical laws are the same in all inertial frames, but our Lorentz velocity transformations are very different to those in classical mechanics for speeds close to the speed of light, because there is now an ultimate speed limit. e.g if we have a constant acceleration, a, then classically we have $v = u + at$ so the speed increases linearly with time WITHOUT limit.

Classical momentum is $p = mv$ so this likewise increases linearly, and $KE = mv^2/2 = p^2/2m$ increases quadratically. But after some time $v \to c$ so our classical definitions of momentum and KE likewise tend to a constant. Yet we are still pouring energy in to accelerate the particle. So what has happened to physical laws of conservation of energy and momentum! how do we make these look the same in all inertial frames?

2.1 Relativisitic momentum

Set up some collisions and analyse them to see that the velocity transformations imply that momentum is not as in classical mechanics $\vec{p} = m\vec{v}$ but is instead $\vec{p} = \gamma(v)m\vec{v}$ where m is proper (or rest) mass of a particle measured in its rest frame and $\gamma(v) = (1 - v^2/c^2)^{-1/2}$ and $v = |\vec{v}|$.

Example: an oil tanker of mass 100kT is travelling at 0.3 m/s. how fast must a 1g hummingbird fly to have the same momentum

The tanker is going very slowly so we can use Newtonian expressions $p_{tanker} \approx mv = 100 \times 10^6 \times 0.3 = 3 \times 10^7$ Ns

What about the bird - if we used newtonian we'd get $(mv)_{bird} = (mv)_{tanker}$ so $v_{bird} = v_{tanker} m_{tanker}/m_{bird} = 0.3 \times 10^8/10^{-3} = 3 \times 10^{10} = 100c$

This shows us we need to use the relativisitic expression $p_{bird} = \gamma(v)mv = m\beta c/\sqrt{1-\beta^2} = 3 \times 10^7$

$\beta/\sqrt{1-\beta^2} = 0.1/1e - 3 = 100$ and $\beta^2 = 10^4/(10^4 + 1)$ so $\beta = 0.99995c$

Example: At what speed does the Newtonian expression for momentum give an error of 5%

The difference between Newtonian and relativistic p is γ so when $\gamma = 1.05$ we get a 5% difference in momentum

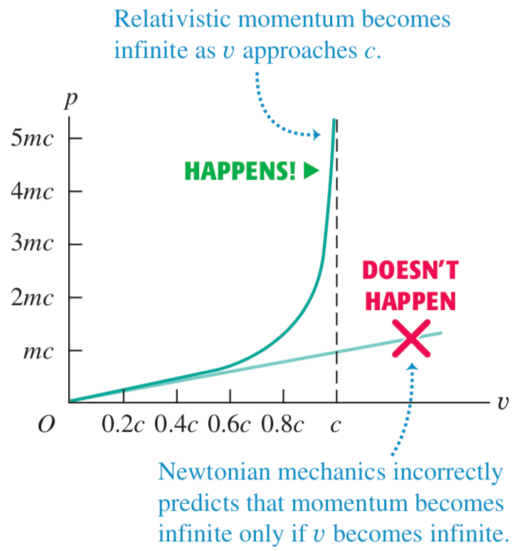

Figure 8:

$1/\sqrt{1-\beta^2} = 1.05$ so $\beta^2 = 1 - 1/1.05^2 = 0.0929$ so $\beta = 0.30c$