2.2 Relativistic Force

Now we have momentum defined properly we are good to go. We can get directly to force via the old classical mechanics $\vec{F} = d\vec{p}/dt = d/dt[\gamma(v)m\vec{v}]$ (where time is measured in the same frame as momentum is measured).

Special relativity can handle accelerations even though its set up for inertial frames. We can always define an instantaneous inertial frame, so accelerating objects moves in continuous fashion from one instantaneous inertial frame to another.

Suppose $\vec{F}$ and $\vec{v}$ are both along the x-axis so the force is accelerating the particle along the direction of motion, increasing its velocity along the x-axis.

$$F_x = \frac{d\gamma}{dt}mv_x + \gamma m\frac{dv_x}{dt} = mv\frac{d}{dt}\frac{1}{(1-v_x^2/c^2)^{1/2}} + \gamma m a_x$$

we need to chain rule, and do a lot of algebra (or type into mathematica) to get $F_x = \gamma^3 m a_x$

if instead $\vec{F}$ and $\vec{v}$ are perpendicular then we get something different. $\vec{F}$ acts perpendicular to $\vec{v}$ so causes the particle to go round in a circle rather than increasing its velocity. so v is constant in magnitude, so $d\gamma/dt = 0$. then we get $\vec{F} = \gamma m\vec{a}$.

example YF37.9 An electron (mass 9.11e-31kg and charge -1.6e-19C) is moving opposte to an electric field of magnitude $E = 5 \times 10^5$ N/C. Find the magnitude of momentum and acceleration at the point when $v = 0.01c, 0.9c$ and $0.99c$.

$p = \gamma mv$ so the different v's imply different γ's of 1.0005, 2.29 and 7.09. hence these velocities have momenta $2.73 \times 10^{-23}, 5.64 \times 10^{-22}, 1.92 \times 10^{-21}$ kg/m/s

acceleration - this is acting in the same direction as the velocity (opposite but changing velocity so we have to consider the $d\gamma/dt$ term. so then we are using $F = \gamma^3 m\vec{a}$ and the force is given by the field $|F| = |q|E = 8 \times 10^{-14}$ N. Hence acceleration $|a| = F/(\gamma^3 m)$ so for each velcoity this is $|a| = 8.8 \times$

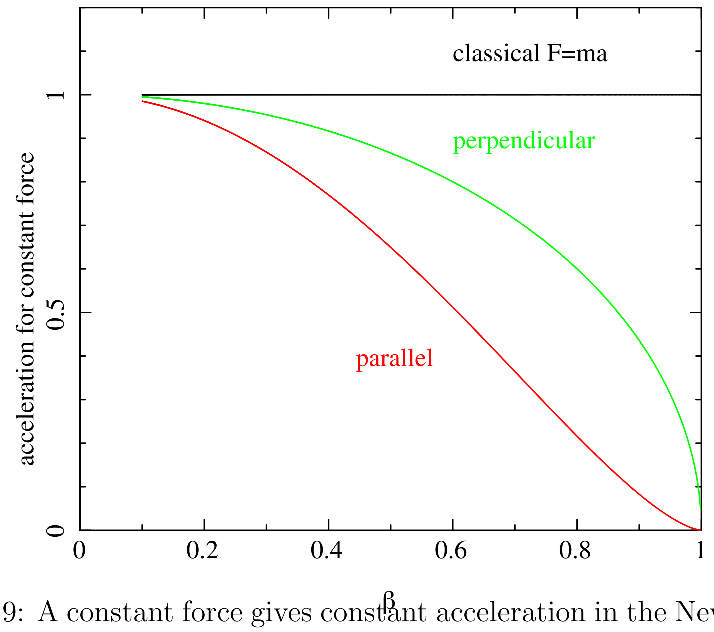

Figure 9: A constant force gives constant acceleration in the Newtonian limit as $F = ma$. But with relativity then if the force is parallel to the direction of motion you get the red line, where $a \propto \gamma^{-3}$ whereas if its perpendicular you get the green line, with $a \propto \gamma^{-1}$.

$10^{16}, 7.3 \times 10^{15}, 2.5 \times 10^{14}$ m/s/s so we can see that the same force does NOT give rise to the same acceleration - which is as expected as we can't go faster than c!

This was all in the lab frame - where the force is constant. if we swap frames and sit on the electron instead then the force isn't constant!!

And if instead this force had been perpendicular to the velocity we would see no change in speed, but there is an acceleration which changes direction. now its $|a| = F/(\gamma m)$ so its $8.8 \times 10^{16}, 3.8 \times 10^{16}, 1.2 \times 10^{16}$ m/s/s.

2.3 Relativisitic energy

So now lets try to get to an understanding of energy in a relativisitc context. We know that energy is force times distance, so when we are accelerating from rest with the force in the same direction as the direction of motion then the work done in going from 0 to some distance x is

$$W = \int_0^x F dx = \int_0^x m\gamma^3 a dx$$

We can break this up as $a_x dx = dv_x/dt dx = dv_x dx/dt = v_x dv_x$

$$W = \int_0^x m\gamma^3 a dx = \int_0^v m\gamma^3 v_x dv_x = m \int_0^v \frac{v_x}{(1 - v_x^2/c^2)^{3/2}} dv_x$$

assuming that we start at from rest at $x = 0$.

lets do a subsitution with $\alpha = 1 - v_x^2/c^2$ to make this integral nicer. then differentiate to get $d\alpha = -2v_x dv_x/c^2$ so $v_x dv_x = -\frac{1}{2}c^2 d\alpha$. We have to remember to change the integral limits as well, so the lower limit is $v_x = 0$ which gives $\alpha = 1$, while the upper limit is whatever α corresponds to the end velocity we accelerate too. Hence

$$W = -\frac{m}{2} \int_1^\alpha c^2 \alpha^{-3/2} d\alpha$$

$$= \frac{-mc^2}{2} \Big[\frac{\alpha^{-1/2}}{-1/2}\Big]_1^\alpha$$

$$= mc^2(\alpha^{-1/2} - 1) = mc^2\Big(\frac{1}{(1 - v^2/c^2)^{1/2}} - 1\Big) = mc^2(\gamma - 1)$$

The relativisitic kinetic energy required to accelerate something from rest to velocity v is $K = mc^2(\gamma - 1)$. Lets look at this in the limits - when the particle is a rest we get KE=0 as expected. then for $v \ll c$ we get an expansion

$$(\gamma - 1) = (1 - \beta^2)^{-1/2} - 1 = 1 + (-1/2)(-\beta^2) \ldots - 1 = v^2/(2c^2)$$

so then $K \approx mc^2 v^2/(2c^2) = 1/2mv^2$ which is the classical value. But as $v \to c$ then the relativistic and classical KE diverge. relativisitic kinetic energy is properly given as $(\gamma - 1)mc^2$ - but this is the difference between two terms, $\gamma mc^2 - mc^2$. The second term exists even when the particle is at rest. This is the rest energy of the particle. We can define total energy $E = K + mc^2 = \gamma mc^2$

So we can accelerate and accelerate and the KE goes up and up. but the speed cannot go faster than the speed of light.