Introduction to modern physics: Special Relativity

March 18, 2020

I'm Prof Chris Done and I get to teach you this section of the course termed 'modern physics'. I got really excited about teaching this course. It was always my favourite bit of being an L1 tutor, that part where physics goes strange! Up till now, what you've seen has been very much built with the physical intuition you get from just looking at the world around you. But all that hard won intuition is not going to help you here, where we extend into the unfamiliar world - the world of the very fast which is special relativity, and the world of the very small which is quantum mechanics. Its not that your physical intution is wrong - it works great for the slow and macroscopic. it just needs extending when we move to more extreme environments.

What I'm going to do in both sections of this course is take you through how physics works in these more extreme environments. I'll get you to build up new physical intuition by doing problems, calculating what happens. Then when you have some new feeling for how this works, I'll talk about the bigger picture, how to think like a relativist, how to think about a quantum world. But we'll get there via the maths, doing the calculations, and doing some more calculations, so you get to build up experience of these unfamiliar worlds.

Books, as ever in first year is Young and Freedman, website lecture notes as ever are on duo.

1 Special Relativity (chapter 37 in YF)

The key thing in special relativity is knowing who is seeing what. Its all about constructing a reference frame for the problem. We'll do this first in 'standard' classical mechanics - called Newtonian or Galilian, and then think about how to change this once we start to think about travelling at speeds close to the speed of light.

1.1 Reference frames in classical mechanics: YF37.1

We can set up a reference frame using a set of axes (usually cartesian). A key concept is an inertial reference frame. In this frame an observer does not experience any net forces - they are not accelerating. Inertial reference frames move at constant velocity with respect to each other.

Quantities are observed differently in different inertial frames, but absolute motion cannot be detected. They are absolutely equivalent. This is called the 'principle of relativity'. For example if two observers are in inertial frames S and S' where S' moves with velocity u relative to someone in S, then someone is S' sees S move with velocity $-u$. and we can transform between events with coordinates in S' $(x'(t'), y'(t'), z'(t'))$ to coordinates in S $(x(t), y(t), z(t))$. I've said t' for symmetry but 'of course' $t' = t$ in Newtonian Physics.

Note: prime does NOT mean derivative. It is common practice to use notation such as S and S' to denote different frames in relativity, so we will always use df/dt for derivative rather than the shorthand f'

Einstein's Principle of Relativity says that once the laws of physics have been established in one inertial frame, they can be applied without modification in any other inertial frame. Both the mathematical form of the laws of physics and the numerical values of basic physical constants that these laws contain are the same in every inertial frame. So far as concerns the laws of physics, all inertial frames are equivalent

But are they? There is already a really interesting point here. There is no absolute frame of motion for reference IN EMPTY space. But space is

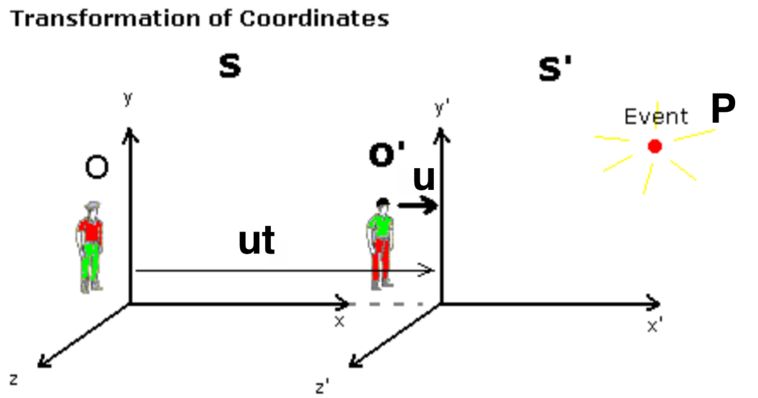


Figure 1:

NOT empty, it is filled with cosmic microwave background radiation. So we CAN define a 'special' inertial frame, one which is at rest compared to the cosmic microwave background. And this can provide an absolute reference frame for the universe, one I can agree on with someone on a planet round a star in Andromeda. But its not special in any other way, its just the frame comoving with the Big Bang expansion. And if I shield my spacecraft from external radiation eg by making it metal so its a Faraday cage so I can't see the cosmic microwave background from inside - then there is no experiement I can do where physics is any different.

Lets make things simple - lets have u point in the directon of the $x - x'$ axis, only do things in 2D, and make the origins coincide at $t = t' = 0$. An event P in S' has initial position $x0', y0'$. If its stationary in S' then an observer, O' at rest in S', sees this at the constant position $x'(t') = x0'$, $y'(t') = y0'$

Another observer, O at rest in S, instead sees this moving and measures coordinates

$$x(t) = x0' + ut \quad y(t) = y0'$$

This is the Newtonian/Galilean world we are used to.

Suppose instead we'd got spacetime coordinates in S and wanted to figure them out in S'. We just replace u with $-u$ and primes with unprimes $x' = x - ut$, $y' = y$, $z' = z$ and of course $t' = t$.

We can use these Newtonian/Galilean coordinate transformations to transform velocities between inertial frames. Suppose there is an object P which is moving with velocity vx', vy' as measured by an observer at rest in S'. Now lets work out what happens to its velocity as seen in S.

$$v_x = \frac{dx}{dt} = \frac{d(x' + ut)}{dt} = v'_x + u$$

$$v_y = \frac{dy}{dt} = \frac{dy'}{dt} = v'_y$$

We can do acceleration also

$$a_x = \frac{dv_x}{dt} = \frac{v'_x + u}{dt} = \frac{dv'_x}{dt} = a'_x$$

$$a_y = a'_y$$

Acceleration is the same even if velocity and position are not, so Newton's laws still work - $F = ma = ma'$.

Newton First law (inertia): if no external forces are acting then an object at rest will remain at rest or if its moving it will continue to move with constant velocity

Newton Second law (acceleration): when an external force acts upon an object it will accelerate in proportion to the magnitude of the net forces and in the direction of that force. The constant of proportionality is the mass so $\vec{F} = m\vec{a}$.

Inertial reference frames are ones in which Newton's first and second laws apply. It follows that general conservation principles of momentum and energy also apply in inertial frames. Any inertial frame moving with constant velocity with respect to another inertial frame is also an inertial frame.

1.2 Approaching light speed

Now lets go faster - an observer sees a spacecraft move past at +1000 m/s. The spacecraft sends out a probe whose speed is 2000 m/s relative to the spacecraft.

We set up the frames so that the spacecraft is in S', where it is at rest. The probe has velocity 2000 m/s relative to the spacecraft so is in this S' frame with velocity $v'_x = 2000$ m/s. The spacecraft frame S' moves with velocity $u = 1000$ m/s relative to an observer O in S.

With Galilean transforms, the observer in S sees the probe as moving at $v_x = v'_x + u = 3000$ m/s.

Instead of launching a probe, the spacecraft turns on a searchlight which travels at speed c relative to the spacecraft. Classical mechanics says the observer in S should see this at speed $v_x = c + u > c$. Maxwells equations says it travels at c.

So something doesn't work. Either there is such a thing as a special inertial reference frame in which Maxwells equations work. Perhaps space is filled with something - an aether - in which electromagnetic waves propagate? Or maybe velocities don't add up the way they should in classical physics. but velocity = distance/time so either distance or time (or both) would have to go very strange!

One way to test this is to look. The Michelson-Morely experiment took the idea of an aether and figured out that the motion of the Earth means that an experiment would move with respect to this aether. so we could detect motion relative to the frame of the aether by looking at the speed of light. This would mean that the laws of physics were NOT the same in all inertial frames, but that there was a special frame for electromagnetism, the frame of the aether.

They measured this with an interferometer, with axes at 90 degrees. The Earth moving with respect to the aether during a year means that they should see a shift in the interference pattern due to the changing speed of light when they rotated it with respect to the aether. They didn't.

1.3 Postulates of special relativity YF37.2-37.3

we saw last lecture that there is no aether/special inertial frame for light. Maxwell is right, light ALWAYS travels at speed c no matter what frame we are in. Its classical physics which is wrong. velocities close to the speed of light DO NOT ADD in the 'standard' Galiean way.

Einstein's reaction to the failure to detect an 'aether' was radical. He embraced it wholeheartedly!

Einstein 1: the laws of physics are the same in every inertial frame of reference (this is in the spirit of Newtonian physics, as we saw from last lecture)

Einstein 2: the speed of light in a vacuum is the same in all inertial frames of reference and is independent of the motion of the source (this takes Maxwells equations at their word! so in some sense we didn't need to say it as its just a specific case of a law of physics in the 1st postulate - but its good just to be explicit because the implications are profound.)

We saw last lecture that the speed of light being constant means that the 'obvious' way in classical mechanics to transform velocties between inertial frames is not correct. Velocity is distance/time so we have to change our definitions of either distance or time or both (its both!)

1.4 light as the ultimate speed limit

The first thing that this tells us is that the speed of light is an ultimate speed limit, that no inertial observer can travel at c. we can see this most easily by doing a thought experiment. Suppose a spacecraft at rest in S' moves with velocity $u = c$ relative to S, and then turns its headlights on. In S, the headlight must travel at c. but so does the spacecraft, so the light is always in the same place as the spacecraft. Yet in the spacecraft in S' they have to see it move away from the spacecraft at c, so they cannot be at the same point in spacetime. As a 16 year old Einstein wondered what he would see if he were travelling at c - but the answer is you can't.

1.5 Simulteneity

A more subtle and disturbing effect of the speed of light being constant in all frames is that events that are simultaneous for one observer need not be simultaneous for another.

Suppose we have multiple lightning strikes as in example YF37.5. Lightning strikes points A' and B' at the back and front of a train carriage and they also burn the ground at A and B. There are two observers, one on the ground, equidistant between A and B and one in the train, equidistant between A' and B'.

These strikes happen simultaneously simultaneously as seen from the observer outside the train.

But the observer in the train is travelling forward. so sees the light from B' before the light from A'. But they know they are equidistant between A' and B', and that light always travels at c. In which case, if A' arrives first then it must have happened first.

Two observers do not need to agree on the order of events which occur at different positions.

A key point here is that SIMULTENEITY is not the same as CAUSALITY. The lightning strike at the front does not cause the lighting strike at the back, so the order in which they happen is not really important. its just a shift in viewpoint.

Its like whether I see a firework explode exactly along my line of sight to the top of the eifel tower - if I was standing somewhere different I'd see it explode to the right or left of the tower. Only if the firework was ATTACHED to the tower would every observer position agrees that it explodes exactly in line with the tower.

whether two events separated in space are simultaneous depends on the motion of the observer. Simulteneity is not an absolute concept (though causality is)

But this means that the time interval between two events is different in

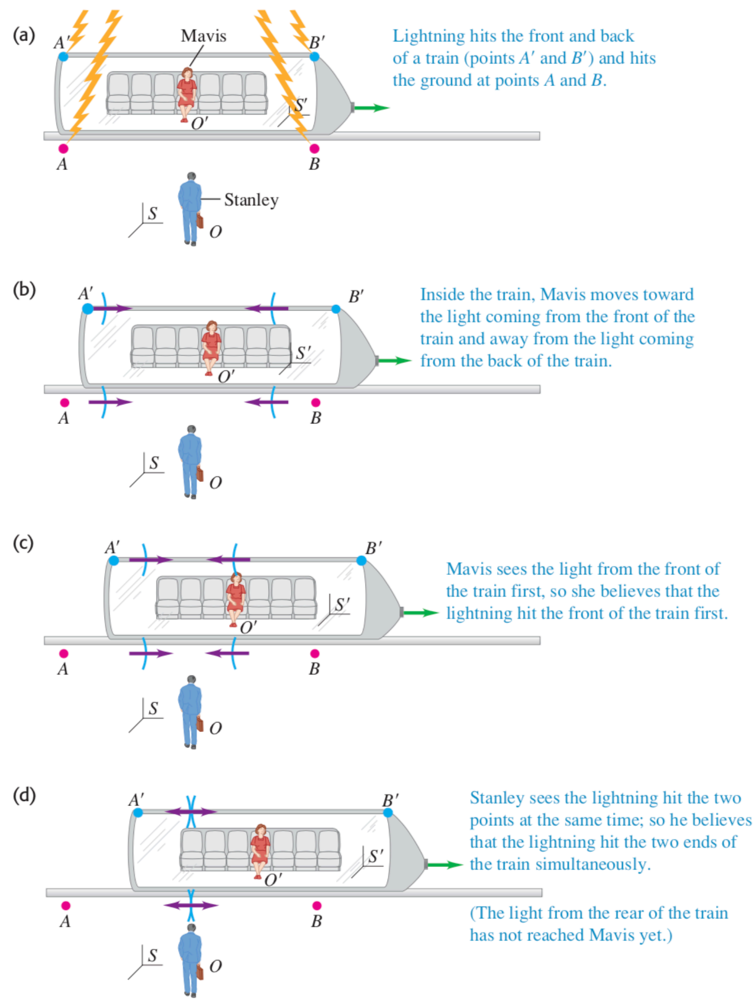
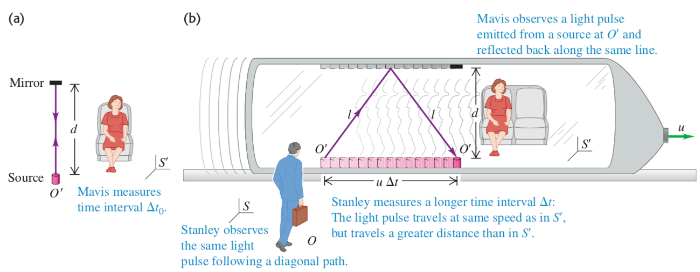

(a)
A′
Mavis
B′
Lightning hits the front and back of a train (points A′ and B′) and hits the ground at points A and B.
S′
O′
A
B
Stanley
S
O
(b)
A′
B′
Inside the train, Mavis moves toward the light coming from the front of the train and away from the light coming from the back of the train.
S′
O′
A
B
S
O
(c)
A′
B′
Mavis sees the light from the front of the train first, so she believes that the lightning hit the front of the train first.
S′
O′
A
B
S
O
(d)
A′
B′
Stanley sees the lightning hit the two points at the same time; so he believes that the lightning hit the two ends of the train simultaneously.
S′
O′
A
B
(The light from the rear of the train has not reached Mavis yet.)
S
O

Figure 2:



37.6 (a) Mavis, in frame of reference S', observes a light pulse emitted from a source at O' and reflected back along the same line. (b) How Stanley (in frame of reference S) and Mavis observe the same light pulse. The positions of O' at the times of departure and return of the pulse are shown.

Figure 3:

different frames.

1.6 Relativity of time intervals - time dilation

We have observers in frame S, and S'. S' moves with velocity u relative to S, along the x-x' direction and their axes O and O' align at t=t'=0.

In S' a beam of light goes vertically upwards from the origin O' and reflects back vertically downwards from a mirror after vertical distance d as measured in S'. In S' the time this takes is $\Delta t' = 2d/c$

In S, the light path is not vertical, but moves horizontally by $u\Delta t$ as well as vertically. so then the path length up to the mirror is ℓ where $\ell^2 = d^2 + (u\Delta t/2)^2$, and the total path length is 2ℓ.

Because light always travels at c this means the time interval measured by someone outside the carriage for the light to get back to the emitter is

$$\Delta t = 2\ell/c = \frac{2}{c}\sqrt{d^2 + \left(\frac{u\Delta t}{2}\right)^2}$$

square this and get

$$\Delta t^2 = \frac{4}{c^2}\left[d^2 + \frac{u^2\Delta t^2}{4}\right] = \frac{4d^2}{c^2} + \frac{u^2}{c^2}\Delta t^2$$

but $\Delta t' = 2d/c$ and $\beta = u/c$ then $\Delta t^2 = (\Delta t')^2 + \beta^2\Delta t^2$ and hence

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}} = \frac{\Delta t'}{\sqrt{1-u^2/c^2}}$$

since $(u/c)^2 < 1$ then $\Delta t > \Delta t'$. The time for the light pulse to go up and down is longer in S than in S'.

Time intervals are measured by clocks. So 'ticks' on a clock as measured in a frame in which the clock is at rest are shorter than those measured in a frame which moves relative to the clock. If you see the clock move, then a clock in your frame measures slower time intervals than the clock in frame moving relative to you. Fast clocks run slow.

The size of this effect is very small in normal life. e.g. $d = 1.5$m and $u = 300$m/s (a plane) then $\Delta t' = 2d/c = 10^{-8}$s inside the plane, whereas someone on the ground measures $\Delta t = \gamma(u)\Delta t'$

$u/c = \beta \ll 1$ so we can use the approximation $(1+x)^n = 1 + nx + \ldots$ where $\gamma = (1-\beta^2)^{-1/2}$ so $x = -\beta^2$ and $n = -1/2$ so $\gamma \approx 1 + (-\beta^2)(-1/2) = 1 + \beta^2/2 = 1 + (300/3\times 10^8)^2/2 = 1 + 10^{-12}/2$. Thus $\Delta t = (1 + 10^{-12}/2)\Delta t'$ and so the fractional difference in times measured is tiny at

$$(\Delta t - \Delta t')/\Delta t' = 10^{-12}/2$$

But now take $u = 0.99c$ then $\gamma = 7.1$ and $\Delta t = 7.1\Delta t'$ and the fractional change is 6.1, or for $u = 0.9999c$, where $\gamma = 70.7$ the fractional change is ~ 70. $\gamma(u)$ asymptotes to ∞ as $u \to c$.

1.6.1 Proper time

There is only one inertial frame in which we are in the same frame as an event, but infinitely many which are moving relative to it. So time intervals

37.8 The quantity $\gamma = 1/\sqrt{1 - u^2/c^2}$ as a function of the relative speed u of two frames of reference.

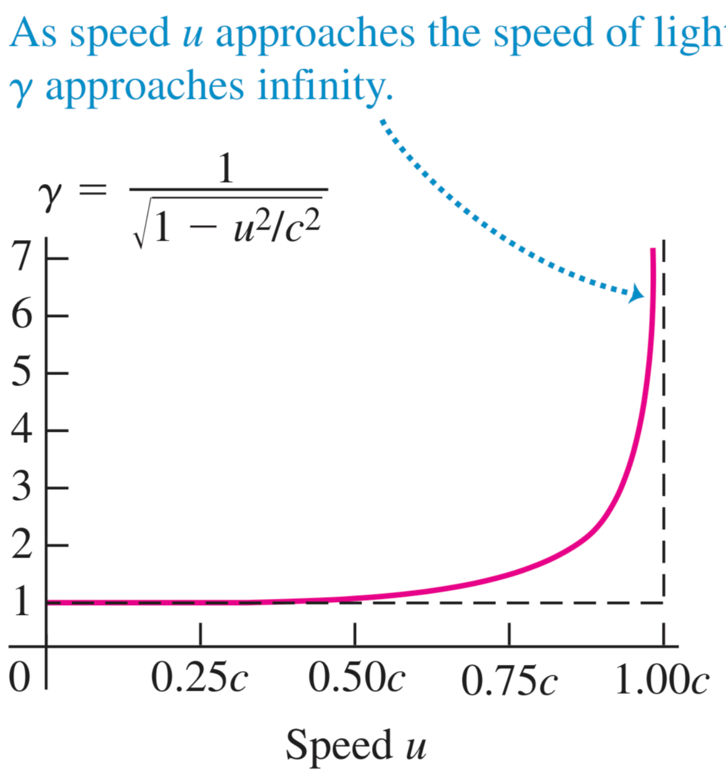


Figure 4:

measured in the same frame as the event have a more fundamental quality than those in any other frame. We use the term **proper time**, T_0, to describe the time interval between 2 events which occur at the same point.

(CAUTION: in the standard setup of S and S', proper time is measured in S' - so you might expect it to be primed, but by convention it does not)

Proper time is always the shortest, all other frames, S, see S' move with velocity u so they measure time intervals which are longer by a factor $\gamma(u)$

1.6.2 Example

e.g. example YF37.1 high energy particles from space interact with atoms in the earths upper atmosphere to produce muons. These decay in their rest frame with lifetime $\Delta t' = 2.2 \times 10^{-6}$ s.

if the muon is moving with respect to the Earth with $u = 0.99c$ then what is its mean lifetime as measured on earth?

$\gamma = 1/\sqrt{1-\beta^2} = (1-0.99^2)^{-1/2} = 7.09$ so $\Delta t = 7.09 \times \Delta t' = 1.56 \times 10^{-5}$ s.

Their half life is $7x$ longer as seen on Earth (a frame in which the muon moves with speed $u = 0.99c$) than experienced in the rest frame of the muon.

1.6.3 Example

Things can get tricky. but generally only because they are described in a confusing way.

Example YF 37.3. A spacecraft with traveller (Mavis) zips past an observer (Stanley) on Earth at $u = 0.6c$. As they pass they both set their clocks to zero. A short time later stanley measures mavis passes a spacestation at a distance stanley measures to be 9×10^7 m away.

What does stanley;s timer read as mavis passes the space station

what does Mavis's timer read as they pass the space station

The standard axes setup is to put the spacecraft observer/mavis/O' in frame S' and the earth observer/stanley/O in frame S. Set $t' = t = 0 = x = x'$ when the observers pass ie when $O' = O$

Stanley measures distance of 9×10^7 m away and a speed of $u = 0.6c$ so this takes $\Delta t = \ell/u = 9 \times 10^7/(0.6 \times 3 \times 10^8) = 0.5\ s$

In S' these events happen at the same point in space which is their own position - first they line up with O and then they line up with the space station. So O' (mavis) is at rest so measures the proper time, which is always the SHORTEST possible time interval. so we need $\Delta t' = \Delta t/\gamma = 0.4$ s

Now it gets more confusing. Stanley blinks as Mavis flys past. Mavis measures the blink to take 0.4 s. How long does Stanley think this takes?

its very tempting to say 0.5 s but this is NOT correct because its a DIFFERENT pair of events. Now its STANLEY who is at rest with respect to the two events (time taken for Stanley's eyelid to go down and up).

So we should relabel the frames. STANLEY is now in the frame S' in which the event is at rest. and MAVIS is in S which is moving relative to S' with velocity $-u$. but time dilation depends only on u^2 so direction doesn't matter. so then STANLEY is in S' so $\Delta t' = \Delta t/\gamma = 0.4/1.25 = 0.32$ s.

This gets us nicely back to our original discussion of simulteneity. Mavis passes the space station after 0.4 s, and sees Stanley finishing his blink. Yet to Stanley, he thinks his blink ends after 0.32 s, way before mavis passes the space station at 0.5 s.

But these events are not causally related so the order in which we see them is not important.

1.6.4 Twin paradox

But surely we can make this causal. Take twins eartha and astrid. Astrid flys away from earth at high speed u, so all time intervals in her rest frame are shortest - including heartbeats! so she ages more slowly than eartha.

But since sign doesn't matter she can turn around and come back at the same speed and still age less. but then she can meet up with her twin and then Astrid is younger than Eartha.

But all inertial frames are relative so surely Eartha could say the same - that she has gone away from astrid at $-u$ and then turned around and met up with her. so Eartha should be younger!

Who is correct? when they meet they should be able to tell who is actually younger!

In fact its astrid. The key is that these frames are not symmetric. Astrid has swapped inertial frames whereas Eartha has always been in the same one (its easier to think of it this way than to get tangled with acceleration/deceleration). Astrid is indeed younger, though its clearer to see this by folding in what is happening to space as well as to time.

1.7 Relativity of length

Speed=distance/time. and we've seen that time intervals have distorted because of the fixed speed of light. Now we will see that length does as well.

We measure can length of an object like a car by making marks on a stationary (relative to the car) pavement at the front and back of the car and measuring between them.

If instead the car is moving with respect to the road then we have to make the marks simultaneously to get the true length of the car - if instead we marked the position of the back of the car a bit later than when we measured the front hen we could get negative length as the back is now further forward than the front was when we measured it.

But then length involves marking simultenously the front and back - so we are back into the issue we had where simultaneous is a frame dependent concept (though causality isn't).

1.8 Lengths parallel to motion

Set up our standard frames S and S', so S' is moving with velocity u wrt S. Put a ruler in S', and measure its length in this rest frame as ℓ'. Attach a light to one end, and a mirror to the other. The total distance is go along the ruler and back to the same point is $2\ell'$ in the rest frame of the ruler, and the time interval (proper time as its all in the rest frame of the ruler) between the light signal starting and being recieved is $\Delta t' = \Delta t'_1 + \Delta t'_2$. In the rest frame, $\Delta t'_1 = \Delta t'_2$ as the light trip is symmetric there and back so $c = 2\ell'/\Delta t'$

In frame S we know that the total length that the light has to travel is the length of the ruler in this frame, which is ℓ plus the frame shift. The length to the mirror is then $\ell_1 = \ell + u\Delta t_1$ and it goes at the speed of light so $c = \ell_1/\Delta t_1$ so $c\Delta t_1 = \ell + u\Delta t_1$ and $c\Delta t_1 = \ell/(c-u)$

On the way back we have $\ell_2 = \ell - u\Delta t_2$ and $c = \ell_2/\Delta t_2$ so $c\Delta t_2 = \ell - u\Delta t_2$ so $c\Delta t_2 = \ell/(c+u)$

The total time measured in S is $\Delta t = \Delta t_1 + \Delta t_2$

$$\Delta t = \frac{\ell}{c-u} + \frac{\ell}{c+u} = \frac{2\ell/c}{1-u^2/c^2} = \gamma^2 2\ell/c$$

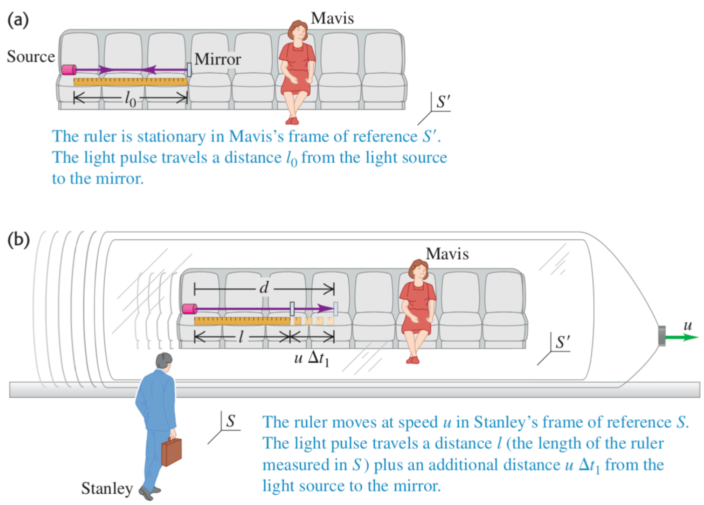


Figure 5:

We also know how the time intervals change from one frame to another $\Delta t = \gamma \Delta t'$

Substitute in and get $\gamma \Delta t' = \gamma^2 2\ell/c$ so $\Delta t' = \gamma 2\ell/c$

In the rest frame we had that $\Delta t' = 2\ell'/c$ so we equate and so $2\ell'/c = \gamma 2\ell/c$ which gives us $\ell = \ell'/\gamma$.

Since $\gamma(u) \geq 1$ then the lengths measured in the moving frame are smaller than lengths measured in the rest frame. This is length contraction. It is real, not an optical illusion, in the same way that time dilation is real - we really age less quickly if we move fast - and we really take up less space if we move fast.

When $u \ll c$ then $\ell \sim \ell'$ and we are back to classical mechanics. However, when $u \to c$ then $\ell \ll \ell'$

We call ℓ' - length measured in the rest frame of the - PROPER DISTANCE ℓ_0 (again beware the lack of primes but it is measured in the primed frame) - in the same way that time intervals measured at a single point are PROPER TIME

Example FY37.4 A spacecraft flies past the earth at a speed of 0.99c. A crew member on board the spacecraft measues its length to be 400m. What length do observers on earth measure?

S' is the frame of the spacecraft. $\ell' = \ell_0 = 400$ m is proper length.

S is earth frame, so $\ell = \ell'/\gamma$ and $\gamma = 1/\sqrt{1-0.99^2} = 7.09$ so $\ell = 400/7.09 =$ 56.4 m

Example YF37.5. (continuing from above) Suppose 2 observers on Earth are 56.4 m apart. How far apart does the spacecraft crew measure them to be?

This is a DIFFERENT EVENT so change frames. Now the EARTH is the rest frame so call it S', and the spacecraft is the new frame, S, that we want to consider. They still have relative velocity u so $gamma = 7.09$ again. But now we are looking at distance in S which is the SPACECRAFT frame relating to a proper distance ℓ' measured in a rest frame so $\ell = \ell'/\gamma = 56.4/7.09$ so $\ell = 7.96$ m

THIS IS NOT THE PROPER LENGTH OF THE SPACECRAFT. As measured on earth it is the length of the spacecraft when the nose and tail are simultaneously measured. In the spacecraft frame these two positions are only 7.96m apart and the nose is 400m in front of the tail. The nose passes O2 before the tail passes O1.

1.8.1 lengths perpendicular to motion

Actually we have already assumed that this didn't change in our discussion of the time transformations! But this should be as at $t = t' = 0$ then the base of the ruler coincided with the x-x' origin. but it lies directly on the y-y' axis. so this lines up at $t = t' = 0$.

1.9 Lorentz transformations

We have an event at point P which is x, y, z, t in frame S and x', y', z', t' in frame S'. who do we relate these coordinates to each other if the origins $O - O'$ are co-incident at $t = t' = 0$ where S' moves relative to S in the +ve x direction at speed u.

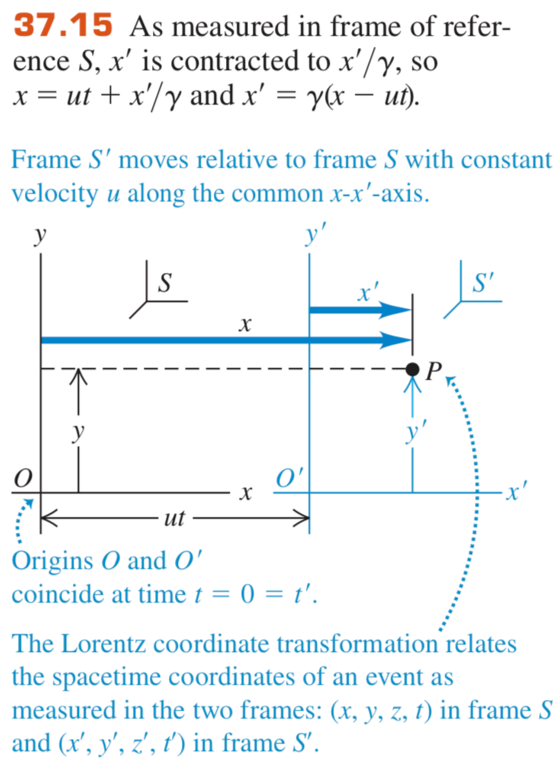


Figure 6:

Lets make it more clear. The distance O' to P is $x' = L'$ is proper distance in S' This will be seen in S as length contracted to $L = L'/\gamma = x'/\gamma$

hence the distance O to P as measured in S is the speed of the frame plus the length contracted distance measured in S i.e.

$x = ut + L = ut + x'/\gamma$

solve for x' and get

$x' = \gamma(x - ut)$

and then we can just write down what happens for S' to S (swap u for -u and primes for unprimes but REMEMBER time

so start at the same place as before but swap to get $x' = -ut' + x/\gamma$

so we have two equations for x' so we can eliminate and solve to get how t relates to t'

$$-ut' + x/\gamma = \gamma x - \gamma ut$$

$$-\gamma ut' + x = \gamma^2 x - \gamma^2 ut$$

$$\gamma ut' = x(1 - \gamma^2) + \gamma^2 ut$$

but we have $1 - \gamma^2 = 1 - 1/(1 - \beta^2) = (1 - \beta^2 - 1)/(1 - \beta^2) = -\beta^2\gamma^2$ so

$$\gamma ut' = \gamma^2 ut - \beta^2\gamma^2 x$$

$$t' = \gamma(t - xu/c^2)$$

bring everything together and we have the Lorentz transformations

$$x' = \gamma(x - ut) \quad y' = y \quad z' = z \quad t' = \gamma(t - xu/c^2)$$

or we can write these for quantities in S given those in S' as (replace $-u$ or u and swap primed/unprimed)

$$x = \gamma(x' + ut') \quad y = y' \quad z = z' \quad t = \gamma(t' + x'u/c^2)$$

For values of $u \ll c$ then $\gamma(u) \to 1$ and we get back to the galilean transformations of classical mechanics. but for $u \to c$ then space and time become intertwined and we can no longer say that they are absolutes independent of the frame of reference. The only SPECIAL frame of reference is the rest frame.

the combination x, y, z, t is called the spacetime coordinates of an event as measured in S. They relate to spacetime coordinates in frame S' via these transformations.

Example YF37.6. Mavis pilots her spacecraft across a finish line at 0.6c relative to the line, winning the race. (event 1). According to her, at the same instant as she, at the front of the ship, sees herself crossing the line, a 'hooray' message is sent from the back of the ship (event 2). She measures the length of her ship to be 300m. Stanley is at the finish line, and is at rest relative to it. When and where does he measure events 1 and 2

Set up axes in the standard way. Mavis is in S' as here she is at rest relative to the ship. let the origins coincide at $t = t' = 0$ when Mavis crosses the line. Then event 1 in S' is at $x'_1, y'_1, z'_1, t'_1 = 0, 0, 0, 0$ and event 2 in S' is at $-300, 0, 0, 0$ as the events are simultaneous in this frame and the ships length is 300m in this frame.

First task is always calculate γ for the problem. $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{(}1 - 0.6^2) = 1.25$

In S event 1 is at $x_1 = \gamma(x'_1 + ut_1) = 0$ and $t_1 = t'_1 = 0$

In S, event 2 is as $x_2 = \gamma(x'_2 + ut_2) = 1.25 \times -300 = -375$ m and is seen at time $t_2 = \gamma(t'_2 + x'_2 u/c^2) = 1.25 \times (0 - 300 \times 0.6/c = -225/c = -7.5 \times 10^{-7}\ s$

And here its negative, so in S, the hooray message comes 0.75μs BEFORE SHE HAS WON!

But its not violating causality because the signal from the back of the spacecraft was not triggered by Mavis. It was an unconnected event.

If instead, Mavis sends a signal to the back of the spacecraft to trigger the 'hurray' then it adds an extra light travel time of $300/c = 1 \times 10^{-6}$s to get from the front to the back of the spacecraft. Hence event 2 in her frame, S', has coordinates $-300, 0, 0, 10^{-6}$

Stanley then measures this new (causal) event 2 happening at time $t_2 = \gamma(t'_2 + x'_2 u/c^2) = 1.25 \times (10^{-6} - 300 \times 0.6/c) = 5 \times 10^{-7}$ s

This is now positive, so causality is actually preserved!

Causality is what we care about because its about a physical connection. Simulteneity is just a question of the observers point of view - observers can disagree about whether an object behind me is seen directly in line with me or projected to my left or my right. its not an important difference, everyones view is 'correct'. but if I am HOLDING the object (so I can affect it physically) then all observers agree that its in line with me.

1.10 Lorentz velocity transformations

We had the Lorentz transformations between coordinates

$$x' = \gamma(x - ut) \quad y' = y \quad z' = z \quad t' = \gamma(t - xu/c^2)$$

but we can re-write them considering only a small displacement dx, dy, dz, dt and get

$$dx' = \gamma(dx - udt) \quad dy' = dy \quad dz' = dz \quad dt' = \gamma(dt - dxu/c^2)$$

$$\begin{aligned} dx'/dt' &= \gamma(dx - udt)/\gamma(dt - dxu/c^2) \\ &= \frac{dx/dt - u}{1 - dx/dtu/c^2} \\ v'_x &= \frac{v_x - u}{1 - v_x u/c^2} \end{aligned}$$

This has some interesting properties! when $u, v_x \ll c$ then $v'_x \to v_x - u$ as expected. but when we go to $v_x \to c$ then $v'_x \to c$ also - lets see this explicitly by setting $v_x = c$

$$v'_x = \frac{c - u}{1 - cu/c^2} = c\frac{c - u}{c - u} = c$$

Anything moving with velocity $\to c$ in S also has velocity $v'_x \to c$ in S' despite the relative notion of the two frames. The speed of light is the same in any frame (by construction)

We get the inverse transforms as ever by swapping primes and unprimes and $-u$ for u.

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2}$$

37.16 The spaceship, robot space probe, and scoutship.

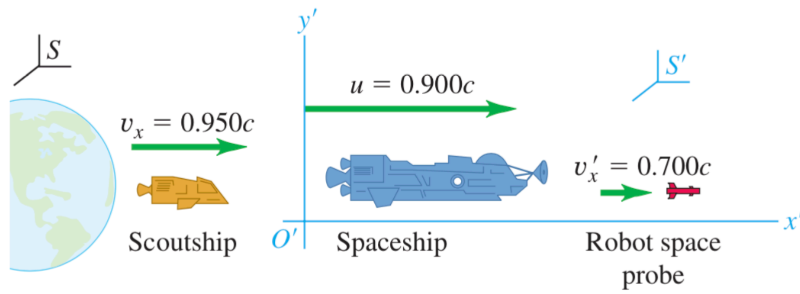


Figure 7:

Example YF37.7 spacecraft moving away from earth at 0.9c fires robot space probe in same direction as its motion at 0.7c as seen from the spacecraft. What is the velocity as seen from the Earth?

Set up the frames. The spacecraft is S', and its velocity relative to earth, S, is $u = 0.9c$.

The robot probe has $v'_x = 0.7c$ - its relative to the spacecraft so its in S'

This would be seen in S as $v_x = \frac{0.7c+0.9c}{1+0.7\times0.9} = 0.982c$

A scoutship is sent from Earth at 0.95c to try to catch up with the spacecraft. What is the speed of the spacecraft with respect to the scout?

We know the scout ship in the Earth frame S has velocity $0.95c$ while the spacecraft has $0.9c$. So the scout is catching up with the spacecraft. We want to measure the scout speed from the spacecraft so the spacecraft should be the one at rest, so $u = 0.9c$

Their relative velocity in frame S' is

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} = c\frac{0.95 - 0.9}{1 - 0.9 \times 0.95} = 0.345c$$

What is the speed of the probe relative to the scout?

We know that the probe is moving at velocity $0.982c$ in the Earth frame. While the scout has $0.95c$ in this frame so the scout will be falling behind. So make S' be the probe rest frame so $u = 0.982c$

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} = c\frac{0.95 - 0.982}{1 - 0.95 \times 0.982} = -0.477c$$

1.11 velocity transformations in orthoganal directions

While the coordinates transformation $y - y'$ and $z - z'$ are not affected by a velocity boost on the x-axis, this is not true for $v_y - v'_y$ and $v_z - v'_z$. This is because time dilation affects motion along all axes.

$$x' = \gamma(x - ut) \quad y' = y \quad z' = z \quad t' = \gamma(t - xu/c^2)$$

but we can re-write them considering only a small displacement dx, dy, dz, dt and get

$$dx' = \gamma(dx - udt) \quad dy' = dy \quad dz' = dz \quad dt' = \gamma(dt - dxu/c^2)$$

$$v'_y = dy'/dt' = \frac{dy}{\gamma(dt - dxu/c^2)} = \frac{dy/dt}{\gamma(1 - dx/dtu/c^2)} = \frac{v_y}{\gamma(1 - v_x u/c^2)}$$

$v'_y \neq v_y$! because even though $dy' = dy$, $dt' \neq dt$ - its the time change between frames which leads to a velocity change!

1.12 Doppler Effect

We could do this by considering lengths and times as in the book. Or we could just do the transformations! Suppose we have an electromagnetic wave. $E'(x', t') = A\cos(k'x' - \omega' t')$ and we know that this travels at $c = \omega'/k' = f'\lambda'$ now do the transformation to frame S:

$E(x,t) \propto \cos[k'\gamma(x-ut) - \omega'\gamma(t-ux/c^2)]$

$= \cos[\gamma(k' + \omega' u/c^2)x - \gamma(\omega' + uk')t]$

but $c = \omega'/k'$ so $ck' = \omega'$ and we have

$= \cos[\gamma(k' + k'u/c)x - \gamma(\omega' + u\omega'/c)t]$

$= \cos[\gamma(1+u/c)k'x - \gamma(1+u/c)\omega' t] = \cos(kx - \omega t)$ where we have k and ω in the S frame. these relate to k' and ω' via the factor

$$\gamma(1+u/c) = \frac{1+u/c}{\sqrt{1-u^2/c^2}}$$

$$= \frac{1+u/c}{\sqrt{(1-u/c)(1+u/c)}} = \sqrt{\frac{1-u/c}{1+u/c}}$$

so we have

$$\omega = \sqrt{\frac{1-u/c}{1+u/c}}\omega' \qquad k = k'\sqrt{\frac{1-u/c}{1+u/c}}$$

and $c = \omega'/k' = \omega/k = (2\pi f)(\lambda/2\pi) = f\lambda$. The wavelength gets longer (frequency gets lower) when its moving away (+u) and shorter (higher frequency) when its moving towards you (-u). But its NOT just a simple γ factor as lengths contract, changing the wavelength as well as time dilating, changing the frequency.

2 Relativisitic kinematics

This all has implications for how we do kinematics, how we define momentum, energy, force - just about everything changes now as $u \to c$.

One of the key assumptions (postulates) was that physical laws are the same in all inertial frames, but our Lorentz velocity transformations are very different to those in classical mechanics for speeds close to the speed of light, because there is now an ultimate speed limit. e.g if we have a constant acceleration, a, then classically we have $v = u + at$ so the speed increases linearly with time WITHOUT limit.

Classical momentum is $p = mv$ so this likewise increases linearly, and $KE = mv^2/2 = p^2/2m$ increases quadratically. But after some time $v \to c$ so our classical definitions of momentum and KE likewise tend to a constant. Yet we are still pouring energy in to accelerate the particle. So what has happened to physical laws of conservation of energy and momentum! how do we make these look the same in all inertial frames?

2.1 Relativisitic momentum

Set up some collisions and analyse them to see that the velocity transformations imply that momentum is not as in classical mechanics $\vec{p} = m\vec{v}$ but is instead $\vec{p} = \gamma(v)m\vec{v}$ where m is proper (or rest) mass of a particle measured in its rest frame and $\gamma(v) = (1 - v^2/c^2)^{-1/2}$ and $v = |\vec{v}|$.

Example: an oil tanker of mass 100kT is travelling at 0.3 m/s. how fast must a 1g hummingbird fly to have the same momentum

The tanker is going very slowly so we can use Newtonian expressions $p_{tanker} \approx mv = 100 \times 10^6 \times 0.3 = 3 \times 10^7$ kg m/s

What about the bird - if we used newtonian we'd get $(mv)_{bird} = (mv)_{tanker}$ so $v_{bird} = v_{tanker}m_{tanker}/m_{bird} = 0.3 \times 10^8/10^{-3} = 3 \times 10^{10} = 100c$!!!

This shows us we need to use the relativisitic expression $p_{bird} = \gamma(v)mv = m\beta c/\sqrt{1 - \beta^2} = 3 \times 10^7$

$\beta/\sqrt{1 - \beta^2} = 0.1/1e - 3 = 100$ and $\beta^2 = 10^4/(10^4 + 1)$ so $\beta = 0.99995c$

Example: At what speed does the Newtonian expression for momentum give an error of 5%

The difference between Newtonian and relativistic p is γ so when $\gamma = 1.05$ we get a 5% difference in momentum

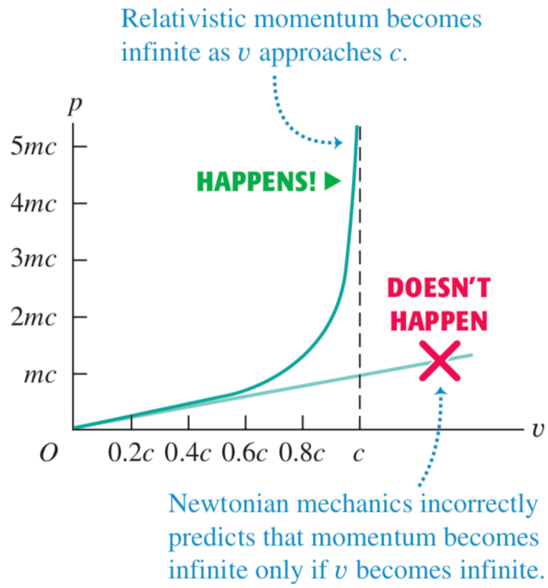


Figure 8:

$1/\sqrt{1-\beta^2} = 1.05$ so $\beta^2 = 1 - 1/1.05^2 = 0.0929$ so $\beta = 0.30c$

2.2 Relativistic Force

Now we have momentum defined properly we are good to go. We can get directly to force via the old classical mechanics $\vec{F} = d\vec{p}/dt = d/dt[\gamma(v)m\vec{v}]$ (where time is measured in the same frame as momentum is measured).

Special relativity can handle accelerations even though its set up for inertial frames. We can always define an instantaneous inertial frame, so accelerating objects moves in continuous fashion from one instantaneous inertial frame to another.

Suppose $\vec{F}$ and $\vec{v}$ are both along the x-axis so the force is accelerating the particle along the direction of motion, increasing its velocity along the x-axis.

$$F_x = \frac{d\gamma}{dt}mv_x + \gamma m\frac{dv_x}{dt} = mv\frac{d}{dt}\frac{1}{(1 - v_x^2/c^2)^{1/2}} + \gamma ma_x$$

we need to chain rule, and do a lot of algebra (or type into mathematica) to get $F_x = \gamma^3 ma_x$

if instead $\vec{F}$ and $\vec{v}$ are perpendicular then we get something different. $\vec{F}$ acts perpendicular to $\vec{v}$ so causes the particle to go round in a circle rather than increasing its velocity. so v is constant in magnitude, so $d\gamma/dt = 0$. then we get $\vec{F} = \gamma m\vec{a}$.

example YF37.9 An electron (mass 9.11e-31kg and charge -1.6e-19C) is moving opposte to an electric field of magnitude $E = 5 \times 10^5$ N/C. Find the magnitude of momentum and acceleration at the point when $v = 0.01c, 0.9c$ and $0.99c$.

$p = \gamma mv$ so the different v's imply different γ's of 1.0005, 2.29 and 7.09. hence these velocities have momenta $2.73 \times 10^{-23}, 5.64 \times 10^{-22}, 1.92 \times 10^{-21}$ kg/m/s

acceleration - this is acting in the same direction as the velocity (opposite but changing velocity so we have to consider the $d\gamma/dt$ term. so then we are using $F = \gamma^3 m\vec{a}$ and the force is given by the field $|F| = |q|E = 8 \times 10^{-14}$ N. Hence acceleration $|a| = F/(\gamma^3 m)$ so for each velcoity this is $|a| = 8.8 \times$

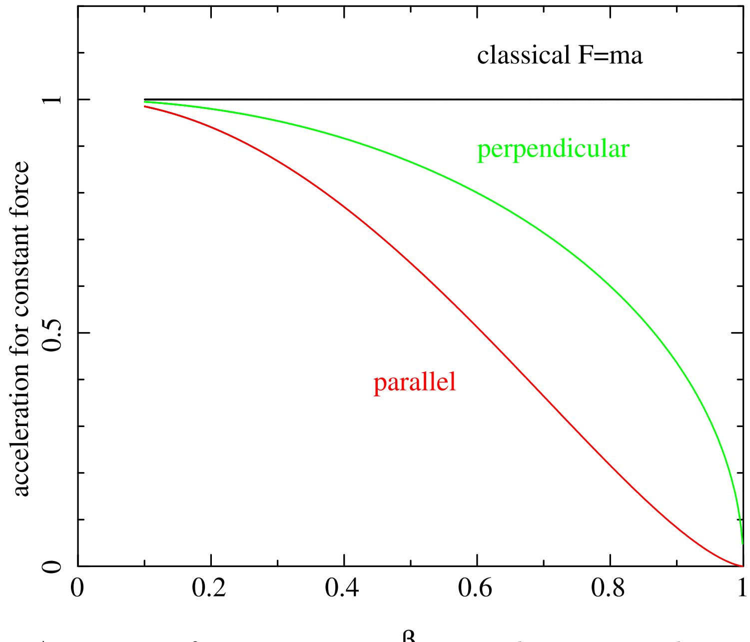


Figure 9: A constant force gives constant acceleration in the Newtonian limit as $F = ma$. But with relativity then if the force is parallel to the direction of motion you get the red line, where $a \propto \gamma^{-3}$ whereas if its perpendicular you get the green line, with $a \propto \gamma^{-1}$.

$10^{16}, 7.3 \times 10^{15}, 2.5 \times 10^{14}$ m/s/s so we can see that the same force does NOT give rise to the same acceleration - which is as expected as we can't go faster than c!

This was all in the lab frame - where the force is constant. if we swap frames and sit on the electron instead then the force isn't constant!!

And if instead this force had been perpendicular to the velocity we would see no change in speed, but there is an acceleration which changes direction. now its $|a| = F/(\gamma m)$ so its $8.8 \times 10^{16}, 3.8 \times 10^{16}, 1.2 \times 10^{16}$ m/s/s.

2.3 Relativisitic energy

So now lets try to get to an understanding of energy in a relativisitc context. We know that energy is force times distance, so when we are accelerating from rest with the force in the same direction as the direction of motion then the work done in going from 0 to some distance x is

$$W = \int_0^x F dx = \int_0^x m\gamma^3 a dx$$

We can break this up as $a_x dx = dv_x/dt dx = dv_x dx/dt = v_x dv_x$

$$W = \int_0^x m\gamma^3 a dx = \int_0^v m\gamma^3 v_x dv_x = m\int_0^v \frac{v_x}{(1-v_x^2/c^2)^{3/2}} dv_x$$

assuming that we start at from rest at $x = 0$.

lets do a subsitution with $\alpha = 1 - v_x^2/c^2$ to make this integral nicer. then differentiate to get $d\alpha = -2v_x dv_x/c^2$ so $v_x dv_x = -\frac{1}{2}c^2 d\alpha$. We have to remember to change the integral limits as well, so the lower limit is $v_x = 0$ which gives $\alpha = 1$, while the upper limit is whatever α corresponds to the end velocity we accelerate too. Hence

$$W = -\frac{m}{2}\int_1^\alpha c^2 \alpha^{-3/2} d\alpha$$

$$= \frac{-mc^2}{2}\Big[\frac{\alpha^{-1/2}}{-1/2}\Big]_1^\alpha$$

$$= mc^2(\alpha^{-1/2} - 1) = mc^2\Big(\frac{1}{(1-v^2/c^2)^{1/2}} - 1\Big) = mc^2(\gamma - 1)$$

The relativisitic kinetic energy required to accelerate something from rest to velocity v is $K = mc^2(\gamma - 1)$. Lets look at this in the limits - when the particle is a rest we get KE=0 as expected. then for $v \ll c$ we get an expansion

$$(\gamma - 1) = (1-\beta^2)^{-1/2} - 1 = 1 + (-1/2)(-\beta^2)\ldots - 1 = v^2/(2c^2)$$

so then $K \approx mc^2v^2/(2c^2) = 1/2mv^2$ which is the classical value. But as $v \to c$ then the relativistic and classical KE diverge. relativisitic kinetic energy is properly given as $(\gamma - 1)mc^2$ - but this is the difference between two terms, $\gamma mc^2 - mc^2$. The second term exists even when the particle is at rest. This is the rest energy of the particle. We can define total energy $E = K + mc^2 = \gamma mc^2$

So we can accelerate and accelerate and the KE goes up and up. but the speed cannot go faster than the speed of light.

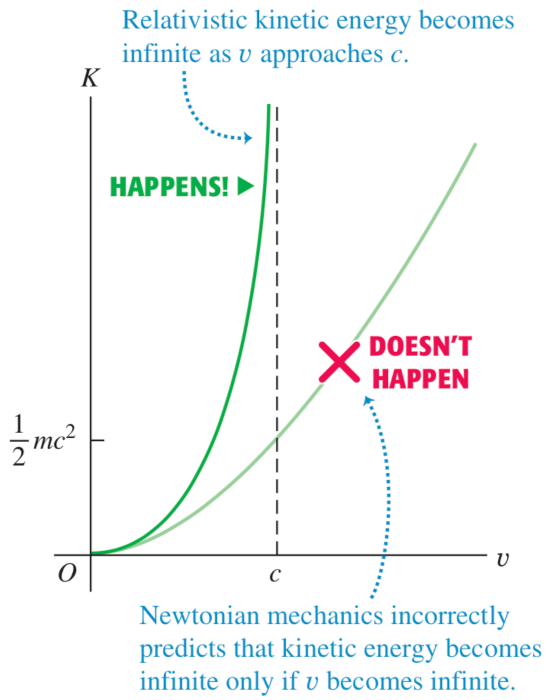


Figure 10:

2.4 Relation between momentum and energy

In the same way that classical KE and momentum can be related via $KE = p^2/2m$ we can particle energy and momentum in relativisitic mechanics as

$E = \gamma mc^2$ and $p = \gamma mv$

re-write these as $E/mc^2 = \gamma = (1-\beta^2)^{-1/2}$ so $(E/mc^2)^2 = 1/(1-\beta^2)$ while $p/mc = \gamma\beta$ so $(p/mc)^2 = \gamma^2\beta^2$

$$\left(\frac{E}{mc^2}\right)^2 - \left(\frac{p}{mc}\right)^2 = \frac{1}{1-\beta^2} - \frac{\beta^2}{1-\beta^2} = \frac{1-\beta^2}{1-\beta^2} = 1$$

$E^2 - p^2m^2c^4/(m^2c^2) = m^2c^4$

$E^2 - p^2c^2 = m^2c^4$ or $E^2 = (pc)^2 + (mc^2)^2$

This implies that a particle with no rest mass can still have energy and momentum - these are photons where $E = pc$.

2.5 Rest mass energy

$E = K + mc^2 = \gamma mc^2$ so even if the particle is at rest it has energy $E = mc^2$ this is energy associated with rest mass rather than with energy of motion

There is clear experiemntal evidence for this

example: Nuclear fusion

$$\frac{1}{1}1H + \frac{7}{3}Li \rightarrow \frac{4}{2}2He + \frac{4}{2}He$$

these have atomic masses of $\frac{1}{1}H = 1.007825$, $\frac{7}{3}Li = 7.016005$ so total before is 8.023830u

$\frac{4}{2}He = 4.002603$ so total after is 8.005206.

We LOST some mass. so we must have GAINED some energy. The energy gained (to KE or the particles) is a net heat of $\Delta Q = 0.018624uc^2$ where $u \approx m_p$.

total energy is $\sim 8uc^2$ so gain in energy is $\Delta Q/Q = 0.018624/8 = 0.002$ i.e. 0.2% of the rest mass energy can be converted to kinetic energy in this reaction. This doesn't sound a lot, but it is actually! Its not far from the most efficience fusion process, which is

$$\frac{1}{1}H + \frac{1}{1}H + \frac{1}{1}H + \frac{1}{1}H \rightarrow \frac{4}{2}He$$

mass before $4 \times 1.007825 =$, afterwards 4.002603. so $\Delta Q = 0.0286uc^2$ and $\Delta Q/Q = 0.007$

This is the reaction which powers the Sun and (most of) the stars we see in the night sky.

MASS NEED NOT BE CONSERVED IN COLLISIONS. The thing we care about is total energy.

example: Neutral pions are unstable particles. they decay to PHOTONS. $\pi^0 \rightarrow \gamma + \gamma$

What is the energy of the photons in the rest frame of the pion?

initial $E_\pi = mc^2$ and $p_\pi = 0$ as it is at rest

final $E = E1 + E2 = h\nu_1 + h\nu_2$ and $|p1| = E1/c$ and $|p2| = E2/c$.

conserve momentum and we need $p1 = -p2$ so $E1 = E2$ - there are two identical photons going in opposite directions,

conserve energy $E = 2h\nu$ and so $2h\nu = m_\pi c^2$

2.6 relativisic collisions and kinematics

We are going to limit ourselves to 1D motion for particles as we've only really done relativity in 1D. we are going to remember that we don't need to conserve mass.

Example: π^0 production: YF example 37.11

Two protons, each with mass 1.67×10^{-27} kg are initially moving in opposite directions. They continue to exist after a head on collisions that produces a neutral pion of mass 2.4×10^{-28} kg. If all particles are at rest after the collision, what is the initial speed of the protons.

momentum before is zero as the particles have same speed but opposite direction.

total energy is conserved. each initial proton has rest energy and kinetic so $E1 = E2 = \gamma(u) m_p c^2$

all particles are at rest afterwards, so only rest mass energy.

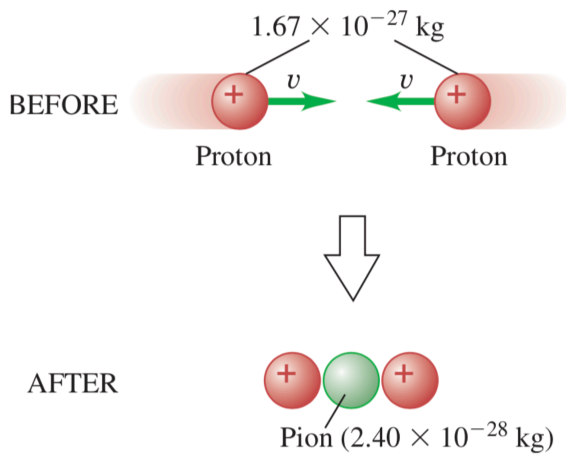


Figure 11: Pion production

$2\gamma_p(u)m_pc^2 = 2m_pc^2 + m_\pi c^2$

$\gamma(u) = (2m_p + m_\pi)/(2m_p) = 1 + m_\pi/(2m_p) = 1.072$

$\gamma = (1-\beta^2)^{-1/2}$ so $\beta^2 = (\gamma^2-1)/\gamma^2$ and $\beta = 0.36$

Example - kaon production! (its a problem in YF37, number is 69 in my version.

some of the incident KE is used to create rest mass energy of the new particles so this is NOT an elastic collision! and it won't conserve mass either!

$$p + p \rightarrow p + p + K^+ + K^-$$

The rest energy of each Kaon is 493.7MeV, the proton is 938.3MeV.

Calculate the minimum kinetic energy for proton 1 which allows this to occur if proton 2 is initially at rest.

This is much easier to do in the centre of mass frame as here the 2 protons have equal and opposite (unknown) velocities of u and $-u$ and EVERYTHING is at rest afterwards.

Conserve energy: before collision $\gamma(u)m_pc^2 + \gamma(-u)m_pc^2 = 2\gamma(u)m_pc^2$

After collision, everything is at rest for a minimum energy collision so $2m_pc^2+$

$2m_K c^2$.

Equate and solve for $\gamma(u) = \gamma(-u)$ and get

$\gamma(u) = 1 + m_K/m_p = 1 + 493.7/938.3 = 1.53$ so $1 - \beta^2 = 1/1.53^2$ and $\beta = 0.76$.

Now we need to transform this to a velocity on one of the protons, so its at rest with respect to the other one.

Pick the one moving to the right, make this S' so we have our standard setup for $u = +0.76c$. in our central frame, we had $v_x = 0.76c$ for the particle on the left, and $v_x = -0.76c$ for the particle on the right.

transform to the primed frame, so we see what the stationary particle sees

particle on the left:

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2} = 0$$

this is as expected as we wanted to be in the rest frame of the particles on the left!

particle on the right - this is what we want!

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2} = -0.76c{-}0.76c/(1{-}(-0.76c)(0.76c)/c^2 = -1.51/1{+}0.57 = -0.96c$$

so the particle on the right has very high velocity in the rest frame of the particle on the left. its kinetic energy $(\gamma - 1)m_p c^2 = (3.68 - 1)m_p c^2 = 2.68.3938MeVc^2 = 2515MeV$ This is much more than the rest mass energy needed to produce the kaons, as in this frame the second proton is moving after the collision.

3 How to think about relativity

Most of this material is now in the new section of FoP1, on Collisions, Conservation and Fields. But I've put it here explicitly as there are some very nice results which also help us understand what special relativity is (and is NOT). its NOT that everything is relative. The order of unconnected events is relative (simultaneous for one observer does NOT mean simultaneous for another), time intervals and lengths depend on how you are moving.... but there are SOME things which are NOT relative. These are conserved quantities - they are invariant and do NOT depend on the frame/motion of the observer.

3.1 Collisions

We saw that for a single particle $E^2 - (pc)^2 = m^2c^4$. and if all we are considering is a single particle then whatever frame we transfer to, m^2c^4 will be the same. so suppose its in a frame where it has velocity βc. then

$$E^2 - (pc)^2 = \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2 c^2 = \gamma^2 m^2 c^4 (1 - v^2/c^2) = m^2 c^4$$

irrespective of the value of β.

But when we have multiple particles involved, what happens next?

In the Relativity homework (Q3, self assessed) you did 2 identical particles, in the centre of momentum frame they had $\beta = \pm 0.41$ ($\gamma = 1.10$) and in the rest frame of one, the other $\beta = 0.7$ ($\gamma = 1.40$).

Centre of momentum frame total energy $E_{tot} = 2\gamma_{CM} mc^2 = 2.2mc^2$ and total momentum is $p_{tot} = 0$ (CM frame!) so $E_{tot}^2 - (p_{tot}c)^2 = 2.2^2 m^2c^4 = 4.81m^2c^4$. This is NOT the sum of the particle masses squared, as thats $4m^2c^4$. But it is still a conserved quantity as you can see from switching to the rest frame of one of the protons, where $E_{tot} = \gamma_E mc^2 + mc^2 = 2.40mc^2$ and $p_{tot} = \gamma m\beta c + 0 = 0.98mc$ but $E_{tot}^2 - (pc)^2 = (2.40^2 + 0.98^2)m^2c^4 = 4.80m^2c^4$ as before!

so this IS an invariant but its NOT the sum of the masses. Its the total energy (rest plus kinetic plus electromagnetic) in the centre of momentum frame.

its obvious that its not mass when we start to think about a neutral pion, decaying into two photons!! the photons are massless, the pion is not.

and the final relativity homework question (Q4) makes this very explicit.

3.2 Invariant interval

There is another, even more fundamental invariant in relativity, which comes from space and time. Its not that everything is relative - there is a sneaky combination which IS absolute, and does not depend on the frame in which they are measured.

Take the Lorentz transformations, and go to the limit of small time and space intervals

$$dx' = \gamma(dx - udt) \quad dy' = dy \quad dz' = dz \quad dt' = \gamma(dt - dxu/c^2)$$

Have a look at $c^2(dt')^2 - (dx')^2 - (dy')^2 - (dz')^2$

$= c^2\gamma^2(dt - dxu/c^2)^2 - \gamma^2(dx - udt)^2 - dy^2 - dz^2$

$= c^2\gamma^2(dt^2 - 2u/c^2 dxdt + u^2/c^4 dx^2) - \gamma^2(dx^2 - 2udxdt + u^2dt^2) - dy^2 - dz^2$

$= c^2dt^2 - dx^2 - dy^2 - dz^2$

So there is some interval which is a combination of time and space which is the same in any frame. what is this interval if we were in the rest frame?

Rest frame is S'. if measuring time between two events at the same place then $dx' = dy' = dz' = 0$ and this interval is $c^2(dt')^2$ where $dt' = T_0 = d\tau$ is proper time!

so we could say $c^2d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

in the rest frame, we also have proper length so if we measure the ends simultaneously we'd get $(dx')^2 + (dy')^2 + (dz')^2 = L_0^2 = ds^2$

so $ds^2 = c^2 d\tau^2 = c^2(dt')^2 - (dx')^2 - (dy')^2 - (dz')^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

where ds is SPACE-TIME interval. It is the same in any frame!

so pull it all together and we have $c^2 d\tau^2 = ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2(dt')^2 - (dx')^2 - (dy')^2 - (dz')^2$ as an invariant

This is called a METRIC it shows how we use coordinates in space and time to determine the distance between any two points. This particular metric, the metric of special relativity, is called the Minkowski metric.

One way to read this is that we have a fixed speed - the speed of light - through SPACETIME. the more of that speed we put through SPACE, the less there is left to travel through TIME, so we travel through time more slowly - we age less.

if you are interested in understanding more try 'The elegant Universe' by Brian Greene

and actually, I learnt some useful stuff from the childrens book series by Russell Stannard, 'The space and time of Uncle Albert' and 'Uncle Albert and the black holes'

One of the amazing insights I got from this was WHY you can't go past the speed of light! we have just done $E = mc^2$ which says mass and energy are interchangeable. so if we have a particle and give it some kinetic energy then in some sense we have added to its inertial mass... and when the kinetic energy starts to dominate over the rest mass then we are in trouble. we accelerate it, which increases its KE, which makes its mass increase, so its harder to accelerate... it gets a bit nasty, as velocity has a direction whereas mass doesn;t so they don't have quite the same properties. but its a helpful way to see that whatever the factor is which scales between mass and energy, then this will be the speed limit.