(a) $A^{\overline{a}}B_{\overline{a}} = \partial x^{\overline{a}}/\partial x^a A^a \partial x^c/\partial x^{\overline{a}} B_c$ [S:2 marks]

$= \delta^c_a A^a B_c = A^a B_a = \chi$ [S:2 marks]

$A^a B_a = 1.5 + 2.4 + 3.3 + 4.2 + 5.1 = 2(5+8) + 9 = 26 + 9 = 35$[S:1 mark]

(b) $D\tau^{ab}/ds = D(\lambda^a \mu^b)/ds = D\lambda^a/ds \mu^b + \lambda^a D\mu^b/ds$ [S:1 mark]

$= \mu^b(d\lambda^a/ds + \Gamma^a{}_{dc}\lambda^d \dot{x}^c) + \lambda^a(d\mu^b/ds + \Gamma^b{}_{dc}\mu^d\dot{x}^c)$ [S:1 mark]

$= \mu^b d\lambda^a/ds + \lambda^a d\mu^b/ds + (\Gamma^a{}_{dc}\lambda^d\mu^b + \Gamma^b{}_{dc}\lambda^a\mu^d)\dot{x}^c$

$= d(\lambda^a\mu^b)/ds + (\Gamma^a{}_{dc}\lambda^d\mu^b + \Gamma^b{}_{dc}\lambda^a\mu^d)\dot{x}^c$ [S:1 mark]

$= d\tau^{ab}/ds + (\Gamma^a{}_{dc}\tau^{db} + \Gamma^b{}_{dc}\tau^{ad})\dot{x}^c$ [S:1 mark]

$= \partial\tau^{ab}/\partial x^c dx^c/ds + (\Gamma^a{}_{dc}\tau^{db} + \Gamma^b{}_{dc}\tau^{ad})\dot{x}^c$

$= (\partial\tau^{ab}/\partial x^c + \Gamma^a{}_{dc}\tau^{db} + \Gamma^b{}_{dc}\tau^{ad})\dot{x}^c = \tau^{ab}{}_{;c}\dot{x}^c$ [S:1 mark]

(c) $r = 1$ and $\phi = 2$

$$R^1{}_{212} = \partial_1\Gamma^1{}_{22} - \partial_2\Gamma^1{}_{21} + \Gamma^e{}_{22}\Gamma^1{}_{e1} - \Gamma^e{}_{21}\Gamma^1{}_{e2}$$

[S:1 mark]

$\partial_1\Gamma^1{}_{22} = \partial_r(-r^3) = -3r^2$

$\partial_2\Gamma^1{}_{21} = \partial_\phi(0) = 0$ [S:1 mark]

$\Gamma^e{}_{22}\Gamma^1{}_{e1} = \Gamma^1{}_{22}\Gamma^1{}_{11} + \Gamma^2{}_{22}\Gamma^1{}_{21}$

$= -r^3. - 1/r + 0 = r^2$ [S:1 mark]

$\Gamma^e{}_{21}\Gamma^1{}_{e2} = \Gamma^1{}_{21}\Gamma^1{}_{12} + \Gamma^2{}_{21}\Gamma^1{}_{22}$

$= 0 + 1/r. - r^3 = -r^2$ [S:1 mark]

total $-3r^2 - 0 + r^2 - -r^2 = -r^2$ [S:1 mark]

(d) local geodesic frame. $R^d{}_{abc;e} = \partial_e(\partial_b\Gamma^d{}_{ac} - \partial_c\Gamma^d{}_{ab})$ [S:1 mark]

perturb indices $bc; e \to ce; b$ $R^d{}_{ace;b} = \partial_b(\partial_c\Gamma^d{}_{ae} - \partial_e\Gamma^d{}_{ac})$

perturb indices $ce; b \to eb; c$ $R^d{}_{aeb;c} = \partial_c(\partial_e\Gamma^d{}_{ab} - \partial_b\Gamma^d{}_{ae})$ [S:1 mark]

add - and remember that order of partial differention doesn';t matter [S:1 mark]

$$R^d{}_{abc;e} + R^d{}_{ace;b} + R^d{}_{aeb;c} = \partial_e(\partial_b\Gamma^d{}_{ac} - \partial_c\Gamma^d{}_{ab})$$

$$+\partial_b(\partial_c\Gamma^d{}_{ae} - \partial_e\Gamma^d{}_{ac}) + \partial_c(\partial_e\Gamma^d{}_{ab} - \partial_b\Gamma^d{}_{ae})$$

$$= \partial_e\partial_b\Gamma^d{}_{ac} - \partial_e\partial_c\Gamma^d{}_{ab} + \partial_b\partial_c\Gamma^d{}_{ae} - \partial_b\partial_e\Gamma^d{}_{ac} + \partial_c\partial_e\Gamma^d{}_{ab} - \partial_c\partial_b\Gamma^d{}_{ae} = 0$$

[S:1 mark]

This is a tensor equation - if it holds in one frame it holds in all frames. [S:1 mark]

(e) $V^2(r) = (1-2m/r)(1+L^2/r^2) = 1-2m/r+L^2/r^2-2mL^2/r^3$ [S:1 mark]

1 is rest mass energy, $1/r$ term comes from normal gravity, $1/r^2$ is normal angular momentum barrier but the last term is a purely GR effect. [S:1 mark]

sketch of the 4 terms giving minimum and maximum (see fig next page) [S:1 mark]

sketch of the minimum and maximum merging (see fig next page) [S:1 mark]

no angular momentum barrier at this point so no stable minimum [S:1 mark]

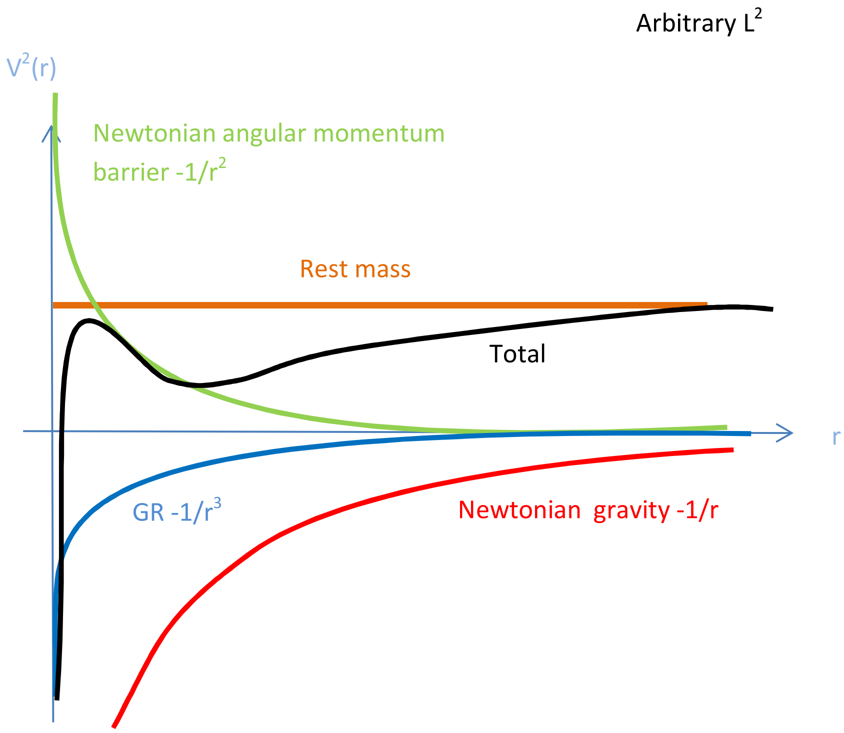
(f) at $r = 2m$ get infinite acceleration on stationary particle [S:1 mark]

spacecraft takes infinite time to reach this as seen by observer far from the hole, so it can never get beyond $r = 2m$ [S:1 mark]

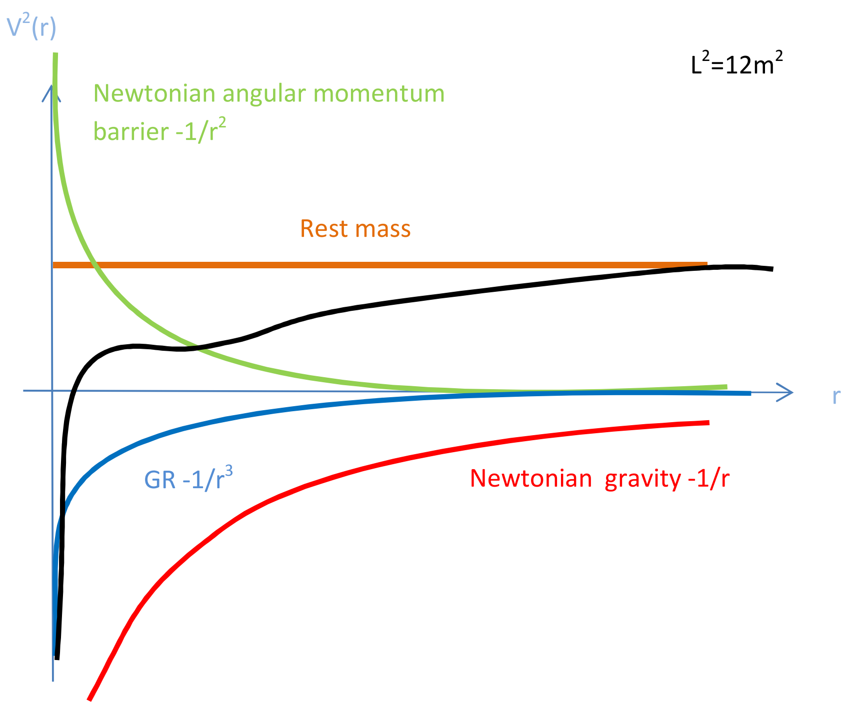
but to someone in the spacecraft they get to $r = 2m$ and below in a finte time [S:1 mark]

metric $ds^2 < 0$ for $r < 2m$ so no real paths - stationary observers don't exist for $r < 2m$? [S:1 mark]

true curvature singularity is at $r = 0$ not 2m. [S:1 mark]



Arbitrary L2
V2(r)
Newtonian angular momentum barrier -1/r2
Rest mass
Total
r
GR -1/r3
Newtonian gravity -1/r



V2(r)
L2=12m2
Newtonian angular momentum barrier -1/r2
Rest mass
r
GR -1/r3
Newtonian gravity -1/r

(a) $L = \frac{1}{2}[(1-2m/r)c^2\dot{t}^2 - r^2\dot{\phi}^2]$ [S:1 mark]

radial E-L is $\partial L/\partial r = \frac{1}{2}c^2\dot{t}^2. - 1. - 2m/r^2 - r\dot{\phi}^2 = c^2\dot{t}^2 m/r^2 - r\dot{\phi}^2$ [S:1 mark]

$\partial L/\partial \dot{r} = 0$ so E-L $\partial L/\partial r = 0$ [S:1 mark]

sub in to get $c^2\dot{t}^2 m = r^3\dot{\phi}^2$ [S:1 mark]

back into metric $c^2 = (1-2m/r)c^2 r^3\dot{\phi}^2/(mc^2) - r^2\dot{\phi}^2$ [S:1 mark]

$mc^2 = (1-2m/r)r^3\dot{\phi}^2 - r^3(m/r)\dot{\phi}^2$ so $\dot{\phi}^2 = mc^2/[r^3(1-3m/r)]$ [U:1 mark]

$\dot{t}^2 = r^3\dot{\phi}^2/(mc^2) = 1/(1-3m/r)$ [U:1 mark]

so $\dot{t} = 1/(1-3m/r)^{1/2}$ and $\dot{\phi} = \sqrt{mc^2/[r^3(1-3m/r)]} = \Omega\dot{t}$ [S:1 mark]

for one orbit $\Delta\phi = 2\pi$ so $2\pi/\tau_{orb} = \sqrt{mc^2/[r^3(1-3m/r)]}$ and so $\tau_{orb} = 2\pi\sqrt{[r^3(1-3m/r)]/(mc^2)}$ [S:2 marks]

(b)
$$\frac{d\lambda^0}{d\tau} + \Gamma^0_{\beta\gamma}\lambda^\beta u^\gamma = \frac{d\lambda^0}{d\tau} + \Gamma^0_{\beta 0}\lambda^\beta \dot{x^0} + \Gamma^0_{\beta 3}\lambda^\beta\dot{\phi} = 0$$

$$\frac{d\lambda^0}{d\tau} + {\Gamma^0}_{10}\lambda^1\dot{x^0} = 0$$

[U:2 marks]

$$\frac{d\lambda^1}{d\tau} + \Gamma^1_{\beta\gamma}\lambda^\beta u^\gamma = \frac{d\lambda^1}{d\tau} + \Gamma^1_{\beta 0}\lambda^\beta \dot{x^0} + \Gamma^1_{\beta 3}\lambda^\beta\dot{\phi} = 0$$

$$\frac{d\lambda^1}{d\tau} + {\Gamma^1}_{00}\lambda^0\dot{x^0} + {\Gamma^1}_{33}\lambda^3\dot{\phi} = 0$$

[U:2 marks]

$$\frac{d\lambda^2}{d\tau} + \Gamma^2_{\beta\gamma}\lambda^\beta u^\gamma = \frac{d\lambda^2}{d\tau} + \Gamma^2_{\beta 0}\lambda^\beta \dot{x^0} + \Gamma^2_{\beta 3}\lambda^\beta\dot{\phi} = 0$$

$$\frac{d\lambda^2}{d\tau} = 0$$

[U:1 mark]

$$\frac{d\lambda^3}{d\tau} + \Gamma^3_{\beta\gamma}\lambda^\beta u^\gamma = \frac{d\lambda^3}{d\tau} + \Gamma^3_{\beta 0}\lambda^\beta \dot{x^0} + \Gamma^3_{\beta 3}\lambda^\beta \dot{\phi} = 0$$

$$\frac{d\lambda^3}{d\tau} + \Gamma^3{}_{13}\lambda^1\dot{\phi} = 0$$

[U:1 mark]

$$\frac{d^2\lambda^1}{d\tau^2} + \Gamma^1{}_{00}\frac{d\lambda^0}{d\tau}\dot{x^0} + \Gamma^1{}_{33}\frac{d\lambda^3}{d\tau}\dot{\phi} = 0$$

sub from 1 and 4

$$\frac{d^2\lambda^1}{d\tau^2} + \Gamma^1{}_{00}\dot{x^0}(-\Gamma^0{}_{10}\lambda^1\dot{x^0}) + \Gamma^1{}_{33}\dot{\phi}(-\Gamma^3{}_{13}\lambda^1\dot{\phi}) = 0$$

[U:2 marks]

$$\frac{d^2\lambda^1}{d\tau^2} = \left(\Gamma^1{}_{00}\Gamma^0{}_{10}(\dot{x^0})^2 + \Gamma^1{}_{33}\Gamma^3{}_{13}\Omega^2\dot{\phi}^2\right)\lambda^1$$

$$\frac{d^2\lambda^1}{d\tau^2} = \left(\frac{m(1-2m/r)}{r^2}\frac{m}{r^2(1-2m/r)}c^2\dot{t}^2 - r(1-2m/r)\frac{1}{r}\dot{\phi}^2\right)\lambda^1$$

$$= \left(\frac{m^2c^2}{r^4}\dot{t}^2 - (1-2m/r)\frac{mc^2\dot{t}^2}{r^3}\right)\lambda^1$$

[U:2 marks]

$$= \frac{mc^2}{r^3}\left(m/r - (1-2m/r)\right)\dot{t}^2\lambda^1 = -\frac{mc^2}{r^3}(1-3m/r)\dot{t}^2\lambda^1$$

$$= -\frac{mc^2}{r^3}(1-3m/r)\frac{1}{(1-3m/r)}\lambda^1 = -\Omega^2\lambda^1$$

[U:2 marks]

this is the equation of SHM so has solution $\lambda^1 = A\cos(\Omega\tau + \phi_0)$ where A, ϕ_0 are constants of integration [U:2 marks]

(c) at start $\tau = 0$ and $\chi(0) = \Omega\tau + \phi_0 = \phi_0$. after one orbit $\chi(\tau_{orb} = \Omega\tau_{orb} + \phi_0$ so difference $\Delta\chi = \Omega\tau_{orb}$ [U:1 mark]

$$= \sqrt{(mc^2)/r^3} 2\pi \sqrt{r^3/(mc^2)} (1 - 3m/r)^{1/2} = 2\pi(1 - 3m/r)^{1/2}$$

[U:2 marks]

binomial approximation $\approx 2\pi(1 - 3m/(2r)) = 2\pi - 3\pi m/r$ [U:1 mark]

parallel transport involves TIME as well as space. so the spatial part precesses but the time part precesses as well and its the total which is parallel transported. [U:2 marks]