4.3 Newtonian gravity meaning of curvature tensor

In Newtonian gravity then two particles initially at rest at the same horizontal distance x but at vertical distance $\pm y$ (where $y \ll x$) from a star of mass M are accelerated radially by the gravitational force $\sim GM/x^2$. Since they are in free fall they are travelling on geodesics, and these geodesics are getting closer together as the particles fall towards the star. Resolving the radial force on one of the stars into a force along the same direction as the other radial force, and one perperdicular to it then there is a force towards the other particle of size $2GM/x^2 y/x$ and the distance apart is $\sim 2y$. Thus there is an acceleration term between the two particles of magnitude which is pulling them together so

$$\frac{d^2y}{dt^2} = -\frac{GM}{x^3}y$$

Convert this now to path length using dirty astrophysicists approximations so $ds^2/dt^2 \sim c^2$

$$\frac{d^2y}{ds^2}\Big(\frac{ds}{dt}\Big)^2 = \frac{d^2y}{ds^2}c^2 = -\frac{GM}{x^3}y$$
$$\frac{d^2y}{ds^2} = -\frac{GM}{c^2x^3}y$$

Compare this with the curvature tensor

$$\frac{D^2\zeta^a}{du^2} = -R^a_{cbd}\zeta^b\dot{x}^c\dot{x}^d$$

Biggest term again is for $c = d = 0$ so

$$\frac{D^2\zeta^a}{du^2} \approx -R^a_{0b0}\zeta^b c^2$$

ζ^a is the separation and its $(0, 0, y, 0)$ as both particles have the same t, x and z then there is only separation in y so this becomes

$$\frac{D^2y}{du^2} \approx -R^y_{0y0}yc^2$$

Again we see explicitally that the tidal forces which we'd associate with gravity can be made simply from curved spacetime. The curvature tensor in some sense tells us about TIDAL forces - how gravity changes from point to point.

And in flat space, then all the $R^a{}_{cbd} = 0$ so $D^2\zeta^a/du^2 = 0$ and geodesics separate as $\zeta^a = Au + B$ i.e. in flat space, geodesics either maintain a constant separation (parallel lines) or separate/approach linearly. So with no gravity things continue in staright lines at constant velocity!

4.4 More on the Curvature tensor

We've seen above that the Riemann curvature tensor is the way to describe curvature in all its gory detail. Hence it also describes other curvature dependent quantities such as the swing in the tensor components if its parallel transported around a little loop about a point P. If the looped components are λ^a(loop) and the old are λ^a then

$$\lambda^a(\text{loop}) - \lambda^a = \Delta\lambda^a = -\frac{1}{2}(R^a{}_{bcd})_P \lambda^b f^{cd}$$

where f^{cd} is a coordinate area term.

Since curvature also comes into the covariant derivative, then its the Riemann curvature tensor which gives the difference in order of derivatives..

$$\lambda^a{}_{;bc} - \lambda^a{}_{;cb} = -R^a{}_{dbc}\lambda^d$$

BUT WHAT IT PHYSICALLY MEANS ISN'T ANY OF THESE!! IT PHYSICALLY MEANS TIDAL FORCES FROM GRAVITY!

4.5 Stress-Energy Tensor

So the curvature of space is related to tidal forces, ie to M/R^3 which is the density of mass. We want to get some tensor equation (valid in any frame) such that curvature=gravity, so we want some tensor equation linking the Riemann curvature tensor and mass density. But it can't just be mass density. Einstein said that mass and energy are equivalent so in fact ALL forms of energy (not just mass) should gravitate. And in fact if this were so it would make some sense of SR. If you accelerate a particle, giving it more and more energy, then as you approach c then the mass increases rather than the velocity. Which is all a bit arbitrary unless all forms of energy gravitate because then as we add energy we increase the inertial (i.e. gravitational) mass. So what we want is a tensor equation linking energy density with curvature.

First we will see how to characterise energy density in SR. In SR we generally dealt with individual particles. where $E = \gamma m_o c^2$ which is the particle energy, $u^\mu = dx^\mu/d\tau$ is 4–velocity, $v^\mu = dx^\mu/dt$ is coordinate velocity and $p^\mu = m_0 u^\mu$ is the 4 momentum of the particle. A stationary particle has $p^\mu = m_0(c, 0, 0, 0)$ while a moving one has $p^\mu = m_0\gamma(c, dx^1/dt, dx^2/dt, dx^3/dt)$ We want to generalise these to look at density. This is a problem because of course density is a frame dependent quantity. The total number of particles in a given volume must be invariant but the volume changes by length contraction along the direction of motion, so if n_0 is the number density in the rest frame then $n_0\gamma$ is the number density as measured in a moving frame. But if we are now moving, then there is a flux of particles across the surfaces. i.e. we take two separate concepts density and flux, and make them into a single 4-vector the number-flux 4-vector $\underline{N} = n_0\underline{u}$ so this has components $N^\mu = n_0 u^\mu$. This is a 4-vector as it transforms in the right way for a contravariant tensor. In the rest frame then $N^\mu = n_0 dx^\mu/d\tau$. In a moving frame then $N^{\nu'} = ndx^{\nu'}/d\tau = n_0 dx^{\nu'}/dx^\mu dx^\mu/d\tau = dx^{\nu'}/dx^\mu N^\mu$. Alternatively we could just have said that we know $u^\mu = dx^\mu/d\tau$ is a 4 vector and so since N^μ is related to it only by invariant quantities (rest frame density) then this must also be a 4-vector.

So that is density done. But we wanted energy density. Lets stick in the rest frame for the time being, then if the particles have no velocity relative to each other then the energy density is simply $m_0 c^2 n_0 = \rho_0 c^2$ where ρ_0 is the mass density in the rest frame. But if we then go to another frame we have to transform the energy AND the density - so it transforms by two factors of γ. So this needs to be a second order contravariant tensor. How about $T^{\mu\nu} = N^\mu p^\nu$ - in the rest frame then this would have only one component $T^{00} = n_0 c m_0 c = \rho_0 c^2$. That sounds like a good start. So $T^{\mu\nu} = N^\mu p^\nu = n u^\mu m_0 u^\nu = \rho_0 u^\mu u^\nu$.

This works if we are only dealing with **dust** i.e. something where the particles have no internal motions, and no stresses or heat conduction or anything complicated. But in general a gas HAS internal motion - particles are moving with respect to each other so there is no rest frame as such for each particle, only a rest frame for the gas as a whole.

we can define a more general $T^{\mu\nu}$ by considering what happens for a

perfect fluid - gas with both rest mass AND internal motions - this is PRESSURE. from SR we know that the energy of this in the rest frame is $3p = 3n_0kT$, so the total energy density in the rest frame is $\rho_0c^2 + 3n_0kT$. So we could do this if

$$T^{\mu\nu} = \mathrm{diag}(\rho_0c^2, p, p, p)$$

pressure is a flux of momentum, so it belongs to the T^{0i} not T^{00} component.

Lets write the stress energy tensor for a perfect fluid as the sum of tensors

$$T^{\mu\nu} = (\rho_0 + p/c^2)u^\mu u^\nu - p\eta^{\mu\nu}$$

where $\eta = diag(1, -1, -1, -1)$ is the flat space (Minkowski) metric tensor i.e. $ds^2 = \eta^{\alpha\beta}dx^\alpha dx^\beta = c^2dt^2 = -dx^2 - dy^2 - dz^2)$ remebering that $x^0 = cdt$. so we can see its a tensor as its made out of things that are either invariant (scalars) or tensors. and in the rest frame it has components $T^{00} = (\rho_0 + p/c^2)c^2 - p = \rho_0c^2$ and $T^{0i} = -p. - 1 = p$ so this indeed has the right form in the rest frame.

its a symmetric tensor $T^{\mu\nu} = T^{\nu\mu}$ as $\eta^{\mu\nu}$ is symmetric and $u^\mu u^\nu = u^\nu u^\mu$.

We can already see something really fun. PRESSURE contributes to energy density, and energy density curves space. So for the interior of a star, as the fuel runs out then the star contracts and gravity gets stronger. so we require a larger pressure to hold the star up. But this presssure adds to the energy density, so increases gravity, so more pressure is needed so .... black holes!

4.6 Conservation laws

The stress energy tensor for a perfect fluid in special relativity.

$$T^{\mu\nu} = (\rho_0 + p/c^2)u^\mu u^\nu - p\eta^{\mu\nu}$$

is a really compact way to count up all the contributions to the energy density, and its a tensor so it works in all (inertial) frames. We did it simply for a perfect fluid (dust+pressure), but it can be made more general as $T^{\mu\nu}$ is defined as the flow of momentum associated with x^μ across the x^ν surface. so T^{00}=energy density, T^{0i}=energy flux through ith surface i.e. heat conduction, T^{i0}=momentum density - if heat is conducted then that energy also has momentum. T^{ij} is shear stress so these are zero for perfect fluid. pressures go perpendicular to surface (no shear stresses).

But the stress energy tensor also embodies more information. Energy is conserved. Rate of change of energy is $l^3\partial T^{00}/\partial t$. But this change is produced by net energy inflow/outflow along the 6 faces of the cube. The energy flow in is $cl^2T^{01}(x)$ and then energy flow out is $l^2cT^{01}(x+l)$ so net inward flow is $l^2(T^{01}(x) - T^{01}(x+l) = -l^2c(\partial T^{01}/\partial x)l = -l^3c\partial T^{01}/\partial x$. There are similar contributions from the other 2 pairs of faces so we have

$$l^3\frac{\partial T^{00}}{c\partial t} = -l^3c\frac{\partial T^{01}}{\partial x} - l^3c\frac{\partial T^{02}}{\partial y} - l^3c\frac{\partial T^{03}}{\partial z}$$

$$\frac{\partial T^{00}}{c\partial t} + \frac{\partial T^{0i}}{\partial x^i} = 0$$

$$\partial_\alpha T^{0\alpha} = 0$$

Momentum is also conserved. and that leads to

$$\partial_\alpha T^{i\alpha} = 0$$

In other words, conservation of both momentum and energy are embodied in

$$\partial_\alpha T^{\beta\alpha} = 0$$

as the stress-energy tensor is symmetric then there is only one divergance $\partial_\alpha T^{\beta\alpha} = \partial_\alpha T^{\alpha\beta}$.

And we can generalise this from SR to GR by replacing partial derivative with covarient derivative (comma goes to semicolon rule) so $T^{\alpha\beta}{}_{;\beta} = 0$

4.7 strong equivalence principle - how to go from SR to GR

Any physical law which can be expressed in tensor notation in SR has EXACTLY the same form in a locally inertial frame of a curve spacetime (GR)! Locally inertial frames (free fall frames) are equivalent to saying we choose to look at a small piece of spacetime, where we can approximate any curved space to a locally flat space. In this choice of coordinates (local cartesian coordinates) then all the christoffel symbols are zero so standard diferentiation and covariant differentiation are the same. BUT once we go outside of this the curvature becomes apparent. So to make our tensor equations work in any general frame we need to swap standard differentiation (which is NOT a real tensor) ($\partial/\partial x^c = \partial_c =_{,c}$ with covariant differentiation $_{;c}$ (which IS a tensor). So this is sometimes called the comma goes to semicolon rule!

and then we need to swap a flat space minkowski metric for a real curved metric. and then we are done!

So if we wanted to generalise $T^{\mu\nu}$ and its conservation laws to GR we'd swap the metric tensor from flat space to generalised curved space so for a perfect fluid

$$T^{\mu\nu} = (\rho_0 + p/c^2)u^\mu u^\nu - pg^{\mu\nu}$$

and take the covariant deriavative rather than the partial derivative

$$T^{\beta\alpha}{}_{;\alpha} = 0$$

4.8 Towards the Einstein equations

Well, we have a bit of a problem. We want a tensor equation which has curvature=energy density. The stress energy tensor is second order,

$$T^{\mu\nu} = (\rho_0 + p/c^2)u^\mu u^\nu - pg^{\mu\nu}$$

this is symmetric as the metric is symmetric. and it also has covariant derivative of zero as this embodies the conservation laws for energy and momentum.

So we are looking for a way to denote curvature which is also a second order symmetric tensor with covariant derivative of zero. But the Riemann curvature tensor which completely embodies all the information about the curvature of space is a 4th order tensor. There is no way that a second order tensor can be equal to a 4th order one. So we need to CONTRACT the Riemann curvature tensor, preferably without losing any information about the spatial curvature!!! and we know from the index symmetries $R_{abcd} = -R_{bacd}$ and $R_{abcd} = -R_{abdc}$ and $R_{abcd} = R_{cdab}$, and $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$ that the Riemann curvature tensor does not have N^4 independent components - it actually has *only*(!) $N^2(N^2-1)/12$. So for a 2D space, there is only one independent component which is R_{1212} when you work it out explicitally.

4.9 Ricci Tensor

If we were to contract R^a_{bcd} we could sum over one of the covariant indices with the contravariant one. But which covariant index - in principle $R^a_{acd} \neq$

$R^a_{bad} \neq R^a_{bca}$.

But the index symmetries $R_{abcd} = -R_{bacd}$ means that $R_{aacd} = -R_{aacd} = 0$ so $R^a{}_{acd} = 0$. So this is not a useful index to contract over!

So now we have the choice of contracting over index 3 or 4. But $R_{abcd} = -R_{abdc}$ so $R^a{}_{bad} = -R^a{}_{bda}$. So modulo a sign change then there is only one non–zero contraction of the Riemann curvature tensor, which we call the **Ricci tensor**.

$$R_{ab} = R^c_{abc}$$

NB there is no widely accepted convention for the sign of the Riemann curvature tensor, or the Ricci tensor, so check the sign conventions of whatever book you are reading.