4.9 Ricci Tensor

If we were to contract R^a_{bcd} we could sum over one of the covariant indices with the contravariant one. But which covariant index - in principle $R^a_{acd} \neq R^a_{bad} \neq R^a_{bca}$.

The index symmetries have some important implications for $R^a{}_{bcd}$. If we are contracting over the first index, $R^a{}_{acd}$ then we can see that $R^a{}_{acd} = g^{ae}R_{eacd} = -g^{ae}R_{aecd} = -g^{ea}R_{aecd} = -R^e{}_{ecd} = -R^a{}_{acd}$. The only way this can be true in general is if $R^a{}_{acd} = 0$. So this is not a useful index to contract over!

So now we have the choice of contracting over index 3 or 4. But $R_{abcd} = -R_{abdc}$ so $R^a{}_{bad} = -R^a{}_{bda}$. So modulo a sign change then there is only one non–zero contraction of the Riemann curvature tensor, which we call the **Ricci tensor**.

$$R_{ab} = R^c_{abc}$$

NB there is no widely accepted convention for the sign of the Riemann curvature tensor, or the Ricci tensor, so check the sign conventions of whatever book you are reading.

The Ricci tensor is a second order tensor about curvature while the stress-energy tensor is a second order tensor about the source of gravity (energy density). So how about $R^{\mu\nu} = \kappa T^{\mu\nu}$ as the equation we are after which is curvature=gravity ? This in fact was Einsteins first guess!

Well, stress energy tensor is also symmetric, and has covariant derivative of zero. The Ricci tensor is also symmetric. Contract the cyclic identity $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$ so

$$R^a{}_{bca} + R^a{}_{cab} + R^a{}_{abc} = 0$$

but $R^a{}_{acd} = 0$ from above so this leaves

$$R^a{}_{bca} + R^a{}_{cab} = 0$$

$$R^a{}_{bca} - R^a{}_{cba} = 0 \text{ from } R_{abcd} = -R_{abdc}$$

$$R_{bc} = R_{cb}$$

So the Ricci tensor is symmetric, as required. Incidentally this means that $R^a{}_b = R_b{}^a$ as $R^a{}_c = g^{ab}R_{bc} = g^{ab}R_{cb} = R_c{}^a$. So we have two out of the 3 qualities we need for the stress energy tensor (second order and symmetric). But what about the third - covariant derivative of zero ?

4.10 Bianchi Identity and Ricci covariant derivative

Again we are going to choose to work in local geodesci coordinates. This defines a flat space in cartesian coordinates tangent to the local curvature at some point. This means the metric is $ds^2 = dx^2 + dy^2$ so all the $\Gamma^a{}_{bc} = 0$. But the derivatives need not be zero!! - we can't transform gravity away in a global sense. IN WHAT FOLLOWS WE ARE LOOKING AT THE COMPONENTS OF THE CURVATURE TENSOR IN THIS SPECIAL LOCALLY INERTIAL FRAME WHERE $\Gamma^a{}_{bc} = 0$

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb}$$

Take the covariant derivative of the Riemann curvature tensor - but in a frame where the Christoffel symbols are zero then this is the same as the normal derivative!

$$R^a{}_{bcd;e} = \partial_e(\partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc}) = \partial_e \partial_c \Gamma^a{}_{bd} - \partial_e \partial_d \Gamma^a{}_{bc}$$

cyclically permuting c, d, e gives

$$R^a{}_{bde;c} = \partial_c \partial_d \Gamma^a{}_{be} - \partial_c \partial_e \Gamma^a{}_{bd}$$

$$R^a{}_{bec;d} = \partial_d \partial_e \Gamma^a{}_{bc} - \partial_d \partial_c \Gamma^a{}_{be}$$

So

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = \partial_e \partial_c \Gamma^a{}_{bd} - \partial_e \partial_d \Gamma^a{}_{bc} + \partial_c \partial_d \Gamma^a{}_{be} - \partial_c \partial_e \Gamma^a{}_{bd} + \partial_d \partial_e \Gamma^a{}_{bc} - \partial_d \partial_c \Gamma^a{}_{be} = 0$$

and again, its a tensor equation, so holds in all frames not just the locally inertial frame.

We can use the Bianchi identity to get this for the Ricci tensor

$$R^a{}_{bca;e} + R^a{}_{bae;c} + R^a{}_{bec;a} = 0$$

$$R_{bc;e} - R_{be;c} + R^a{}_{bec;a} = 0$$

we know (section 3.10) that we can take the metric in and out of covariant differentiation ie that raising and lowering of indices can go in and out of covariant differentiation. So lets raise the index b - multiply it by g^{fb}.

$$g^{fb} R_{bc;e} - g^{fb} R_{be;c} + g^{fb} R^a{}_{bec;a} = 0$$

$$R^f{}_{c;e} - R^f{}_{e;c} + R^{af}{}_{ec;a} = 0$$

contract over e and f ie set $e = f$

$$R^f{}_{c;f} - R^f{}_{f;c} + R^{af}{}_{fc;a} = 0$$

Define $R = R^a{}_a = R_a{}^a$ as the Riemann curvature scalar. Then we have

$$R^f{}_{c;f} - R_{;c} + R^{af}{}_{fc;a} = 0$$

$$R^f{}_{c;f} - R_{;c} + R^{fa}{}_{cf;a} = 0 \quad \text{from } R_{abcd} = -R_{abdc} = R_{badc}$$

$$R^f{}_{c;f} - R_{;c} + R^a{}_{c;a} = 0$$

$$2R^a_{c;a} - R_{;c} = 0$$

$$(R^a_c - \frac{1}{2}R\delta^a_c)_{;a} = 0$$

$$(g^{bc}R^a_c - \frac{1}{2}g^{bc}\delta^a_c R)_{;a} = 0$$

$$(R^{ba} - \frac{1}{2}g^{ba}R)_{;b} = 0$$

so the covariant derivative of the Ricci tensor IS NOT ZERO! But this funny combination of the ricci tensor and curvature scalar IS.

So we have the Ricci Tensor, which is a symmetric second order tensor, but its divergance IS NOT zero. so we cannot write gravity=curvature as $T^{\mu\nu} = R^{\mu\nu}$!!!

ARRGGHHHH!!!

But this funny combination of the ricci tensor and curvature scalar DOES have zero divergance, and IS just a function of curvature. so how about:

$$R^{ba} - \frac{1}{2}g^{ba}R = T^{ba}$$

except of course there can be a scalar relationship between them

$$R^{ba} - \frac{1}{2}g^{ba}R = \kappa T^{ba}$$

and an integration constant:

$$R^{ba} - \frac{1}{2}g^{ba}R + \Lambda g^{ba} = \kappa T^{ba}$$

where Λ is a cosmological constant. This is the Einstein equation i.e. curvature=gravity. Note that we did NOT derive them from first principles. This is simply the lowest order way to write curvature=gravity.

For black holes, the cosmological constant is not important, so we will set $\Lambda = 0$ in what follows. Then we have for 4D spacetime:

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = T^{\alpha\beta}$$

we can recast this in slightly different form:

$$g_{\gamma\alpha} R^{\alpha\beta} - \frac{1}{2} g_{\gamma\alpha} g^{\alpha\beta} R = \kappa g_{\gamma\alpha} T^{\alpha\beta}$$

$$R^{\beta}_{\gamma} - \frac{1}{2} \delta^{\beta}_{\gamma} R = \kappa T^{\beta}_{\gamma}$$

contract over β and γ i.e. set $\beta = \gamma$ Then since $\delta^{\beta}_{\beta} = 4$ then

$$R - 2R = \kappa T$$

$$R = -\kappa T$$

So an alternative form (which we are going to use below) is

$$R^{\alpha\beta} = \kappa (T^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} T)$$