5.8 Particle Orbits in strong field GR

We did wimpy weak field geodesics for particles and photons. Now lets do it in full strong field GR. Lets look at $\dot{r}$ - radial coordinate with respect to (proper) time.

Now lets see what Einstein predicts:

$$ds^2 = c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2 d\phi^2$$

cyclic coordinates $(1 - 2m/r)\dot{t} = E/c^2$ and $r^2\dot{\phi} = L_z$ substitute back into it the metric and solve for $\dot{r}$

$$c^2 = (1 - 2m/r)c^2\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$

$$c^2(1 - 2m/r) = E^2/c^2 - \dot{r}^2 - L_z^2(1 - 2m/r)/r^2$$

$$\dot{r}^2 = E^2/c^2 - c^2(1 + L_z^2/c^2r^2)(1 - 2m/r)$$

$$\dot{r}^2 = E^2/c^2 - V^2(r)$$

where $V^2(r) = c^2(1 + L_z^2/c^2r^2)(1 - 2m/r) = c^2 - 2m/r + L_z^2/r^2 - 2mL_z^2/r^3$. substitute for $m = GM/c^2$ and get

$V^2(r) = c^2 - 2GM/c^2r + L_z^2/r^2 - 2GML_z^2/c^2r^3$.

I called it V^2 as then when we use the binomial expansion $(1+x)^n \approx 1+nx$ then we have

$$V(r)/c = (1+L_z^2/c^2r^2)^{1/2}(1-2GM/c^2r)^{1/2} \approx (1+L_z^2/c^2r^2+...)(1-2GM/2c^2r+...)$$

$$\approx 1/c^2[c^2 - GM/r + L_z^2/2r^2 - GML_z^2/c^2r^3 + \ldots]$$

which makes it obvious that the $GM/(c^2r)$ term is gravitational potential. and the $L_2^2/(2c^2r^2)$ is a centrifugal barrier. - we can see this explicitally by saying centrifugal force is $F = -mv^2/r$. but $L_z = r^2\dot{\phi} = rv$ for circular motion so $F/m = -L_z^2/r^3$. Potential is the integral of force so this gives a potential term of $= L_z^2/2r^2$. But then there is an additional term - the $-GML_z^2/c^2r^3$. this is -ve so it adds to the attractive gravity term, and makes gravity stronger in GR than in newtonian

so to compare directly with Newtonian prediction. Again lets start with total energy $\tilde{E} = \frac{1}{2}v^2 - GM/r$. then $v^2 = (v_r^2 + v_\phi^2) = \dot{r}^2 + r^2\dot{\phi}^2)$ so

$$\tilde{E} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - GM/r$$

$$2\tilde{E} = \dot{r}^2 + r^2 L_z^2/r^4 - 2GM/r$$
$$\dot{r}^2 = 2\tilde{E} - L_z^2/r^2 + 2GM/r$$
$$= 2\tilde{E} - (L_z^2/r^2 - 2GM/r) = c^2[2\tilde{E}/c^2 - (L_z^2/c^2r^2 - 2GM/c^2r)]$$

so its exactly analogous to the V^2 term we had before with $-2GM/c^2r$ represents gravitional potential energy and L_z^2/c^2r^2 is the centrifugal barrier. hence in newtonian we get an effective potential which is centrifugal+gravity. The first term is the angular momentum barrier - particle orbits around a large mass - it wants to get closer because gravity is attractive, but as it gets closer then by conservation of angular momentum it goes faster so there is a bigger force outwards. This $V(r) \to \infty$ as $r \to 0$, and $\to 0$ as $r \to \infty$ and has a minimum. If the particle energy $E = V_{min}$ then the orbit is circular. if $V_{min} < E < 0$ then there is a range of r from r_{min} to r_{max} which the particle can access - this is an elliptical orbit. And if instead $E > 0$ then it is not bounded at large r, but only limited at small r by the angular momentum barrier - this is a hyperbolic orbit.

Hence the full (squared) GR potential is a bit diffferent to the Newtonian shape. $V^2 = c^2(1 - 2m/r + L_z^2/c^2r^2 - 2mL_z^2/c^2r^3)$. so first term is rest mass, second is standard gravity, third is angular momentum barrier. But the 4th is simply different. its -ve so adds to the effect of gravity. gravity is STRONGER in GR than in newtonian.

lets look at the shape in a bit more detail. $V^2(r)$ is zero at $r = 2m$, $\to -\infty$ as $r \to 0$ and $\to 1$ as $r \to \infty$. turning points where $dV^2/dr = 0$.

$$(1 + L_z^2/c^2r^2) - 2md(r^{-1})/dr + (1 - 2m/r)L_z^2/c^2d(r^{-2})/dr = 0$$

$$(1 + L_z^2/c^2r^2)2m/r^2 = 2L_z^2/c^2r^3(1 - 2m/r)$$
$$(1 + L_z^2/c^2r^2)mr^2 = L_z^2r/c^2(1 - 2m/r)$$
$$mr^2 + mL_z^2/c^2 = L_z^2r/c^2 - 2mL_z^2/c^2$$
$$r^2 - L_z^2r/mc^2 + 3L_z^2/c^2 = 0$$
$$r = [L_z^2/mc^2 \pm \sqrt{L_z^4/m^2c^4 - 4.3L_z^2/c^2}]/2$$
$$r = L_z^2/2mc^2 \pm L_z^2/2mc^2\sqrt{1 - 12m^2c^2/L_z^2}$$

So in general has 2 turning points. Do d^2V^2/dr^2 and see that the maximum is at smaller r, minimum at larger r.

But for $L_z = \sqrt{12}mc$ then the two turning points merge together. Put this back into the equation and this happens at $r = 12m^2c^2/(2mc^2) = 6m$. $d^2V^2/dr^2 = 0$ so this is neither a maximum nor a minimum - its a point of inflection.

Stable orbits require a MINIMUM in the potential and that the particle energy E^2 is less than V^2 on either side of the minimum so that the particle is confined to a range of radii (elliptical orbit). If instead the particle energy is exactly at the minimum of the potential then there is no range in radii accessible to the particle and its a circular orbit. If E^2 is such that there is a potential barrier at small r, but NOT at larger r then this is an unbound hyperbolic orbit. If $E^2 > V^2_{max}$ then the particle can get everywhere, even to $r = 0$ and hence it hits the black hole (and never comes out!!)

Anyway, for a point of inflection there is no minimum. so this is NOT a stable orbit. Its unstable - any small perturbation and the particle will spiral in towards the black hole. $r = 6m$ is the last stable circular orbit around a Schwarzschild black hole.

This is a bit different to Newtonian gravity, where there was always a stable orbit possible - you can go closer by orbiting faster. Aha, so maybe this is in fact the point at which we are going around at the speed of light! the event horizon is where the radial velocity becomes equal to the speed of light in order to escape - perhaps this is where our azimuthal orbital speed equals the speed of light. But no, its not, as orbital velocity $v_\phi = r\dot{\phi}$ and $r^2\dot{\phi} = L_z$ as normal so $v_\phi = rL_z/r^2 = L_z/r = \sqrt{12}mc/(6m) = c/\sqrt{3} \sim 0.58c$. so we are not even going round at anything particularly close to the speed of light! Its just a prediction of Einstein;s gravity in the strong field limit.

And we can test this - see slides, as accretion of gas onto a black hole means we get an accretion disc which is a collection of circular orbits. einstein says the inner edge should be at $r = 6m$ and this is OBSERVABLE as the inner edge of the disc is where the maximum gravitational potential gives the maximum temperature emission. and GR looks like it works even in this strong field limit - there is something that looks very like a constant innermost stable circular orbit at which the accretion disc emission stops as the particles fall freely into the black hole.