5.9 Circular orbits: particles

Then our metric simplifies a bit as $dr = 0$

$$ds^2 = c^2 d\tau^2 = c^2(1 - 2m/r)dt^2 - r^2 d\phi^2$$

so the Lagrangian is $L = 1/2(c^2(1 - 2m/r)\dot{t}^2 - r^2\dot{\phi}^2)$ We can do the Euler lagrange equations for r

$L = \frac{1}{2}[(1 - 2m/r)c^2\dot{t}^2 - r^2\dot{\phi}^2]$

$$\frac{\partial L}{\partial \dot{r}} = 0$$

$$\frac{\partial L}{\partial r} = c^2\dot{t}^2 \frac{\partial}{\partial r}(-m/r) - r\dot{\phi}^2 = c^2\dot{t}^2 m/r^2 - r\dot{\phi}^2$$

E-L equation in r is $d/d\tau(0) - (c^2\dot{t}^2 m/r^2 - r\dot{\phi}^2) = 0$

$c^2\dot{t}^2 m/r^2 = r\dot{\phi}^2$

so the COORDINATE time for one orbit is

$$mc^2 t_{orb}^2 = r^3(2\pi)^2 \quad t_{orb} = 2\pi[r^3/(mc^2)]^{1/2} = 2\pi(r^3/GM)^{1/2}$$

ie same as Newtonian except this is COORDINATE time, not proper time.

PROPER TIME measured by someone going round on this orbit (sub. back into metric)

$$c^2 = c^2(1 - 2m/r)\dot{t}^2 - r^2\dot{\phi}^2 = c^2(1 - 2m/r)\dot{t}^2 - r^2 mc^2\dot{t}^2/r^3$$

$$1 = (1 - 3m/r)(dt/d\tau)^2$$

$$d\tau = (1 - 3m/r)^{1/2} dt$$

$$\tau_{orb} = (1 - 3m/r)^{1/2} t_{orb}$$

We saw last lecture that there is a last stable orbit at $r = 6m$. so lets do what happens here. The proper time for one orbit as experienced by the orbiter is

$$\tau_{orb} = (1 - 3m/r)^{1/2} t_{orb} = t_{orb}/\sqrt{2}$$

or we can put numbers in and do

$$t_{orb} = 2\pi[(6m)^3/(mc^2)]^{1/2} = 2\pi 14.7m/c = 92.3m/c$$

hence $\tau_{orb} = 65.3m/c$

proper time for someone INSIDE the spacecraft orbiting at $r = 6m$ is different the proper time as experienced by someone hovering at this radius - held there by firing rockets (not on a geodesic). We know how to relate proper time to coordinate time when we are stationary - its simply the metric with no spatial parts as all the $dr, d\theta, d\phi = 0$ i.e. $d\tau = (1 - 2m/r)^{1/2}dt$.

hence the proper time as measured for one orbit at $r = 6m$ for the hovering craft is $\tau_{hov} = (1-2m/r)^{1/2}t_{orb} = (2/3)^{1/2}t_{orb} = 0.816t_{orb}$ compared to $t_{orb}/\sqrt{2} = 0.707t_{orb}$

The astronauts in the hovering probe see a LONGER elapsed time - they are older than their companions who orbited.

When the orbiting spacecraft comes past, the people in the hovering spacecraft send a radial light signal to friends at ∞. So the proper time interval for one orbit as measured by these at ∞ is $c^2d\tau^2 = (1-2m/\infty)c^2dt^2$ so $d\tau = dt$ - proper time is the same as coordinate time. so for one orbit, someone stationary at infinity thinks it takes a time $\tau_\infty = t_{orb}$ which is the same as the newtonian time for an orbit! then the hovering spacecraft sees it take $0.816t_{orb}$ while the person on the geodesic sees the smallest time at $0.707r_{orb}$. geodesics are minimum proper time as they are minimum proper distance paths.

5.10 Black holes as time machines

well, so we can age at only 70% of the rate as someone at infinity if we are orbiting at $r = 6m$. This isn't really very impressive. if we have rockets we can do better as we can orbit further in, using the rockets to do the path corrections to these unstable circular orbits.

for the orbiting observer, $\tau_{orb} = (1 - 3m/r)^{1/2}t_{orb}$. So proper time goes to zero at $r = 3m$, and goes COMPLEX for $r < 3m$. What does this MEAN physically ? it means that there are NO geodesic circular orbits possible at $r < 3m$. and that the proper time goes to zero for $r = 3m$. that makes us think of light as light travels on null geodesics, puts all its fixed spped c through space so has none left to travel through time.

so we guess that $r = 3m$ is where light orbits. lets check by solving the circular orbit equations again for light.

for light we still have the radial E-L equation $c^2\dot{t}^2m/r^2 = r\dot{\phi}^2$ but the metric is now null so

$$0 = c^2(1-2m/r)\dot{t}^2 - r^2\dot{\phi}^2 = c^2(1-2m/r)\dot{t}^2 - r^2mc^2\dot{t}^2/r^3$$

$$0 = (1-3m/r)(dt/d\tau)^2$$

hence light orbits at $r = 3m$. and if we had done a full $V^2(r)$ from non-circular orbits we'd see that this was an unstable orbit.

This is very different to Newtonian gravity where circular orbits are always possible. In Newtonian orbits you can always balance gravity just by running round faster. In SR/GR you can only run as fast as the speed of light. $r = 3m$ is the orbit for going at the speed of light. So there are no faster orbits. And if you were going around on this orbit, then you wouldn't age as you are going at the speed of light.

so we can use this to make an effective time machine. have twins (labeled 1 and 2) start at some radius $r \to \infty$. Twin 1 goes in a spacecraft and drops down to a radius *just* above $3m$, which they time as taking proper time τ_{drop}. They then do millions of orbits at this radius in a a very small proper time, and then they fire the rockets and go back. Assume their rockets are good enough to make this take the same time as τ_{drop}. Then they have aged $\approx 2\tau_{drop}$, whereas twin 2 has aged by *much* more.

so this is an effective way to travel into the future. But NOT the past!!

5.11 using tensors to calculate redshifts on circular orbits

All this so far was very algebraic - does this mean that we can now drop all that evil tensor maths ? NO! actually, the evil tensors are our friends for whenever things get a bit complicated. Suppose our orbiting spacecraft was sending light signals back out to infinity. and we wanted to know what frequency we observed them to be if the rest wavelength emitted is ν_{em}. we can see that there will be quite a few effects to keep track of. there are the doppler shifts - when the spacecraft is at the tangent point, comming directly towards us, there should be blueshift, and when its at the other tangent point, going directly away, there is a redshift. But on both sides there should also be a special relativisitc time dialiation as 'fast clocks run slow' and a gravitational redshift as well since its being emitted in a strong gravitational field. doing all this correctly is going to be tough. instead, we can just do the tensors as the emitted frequencies are $(u_\alpha p^\alpha)_{em}$ where the 4-velocity is that of the emitting particle and the 4-momentum that of the light, and the observed ones are $(u_\alpha p^\alpha)_{obs}$ where the 4-velocity is the velocity of the observer and 4-momentum is that of the light they observe. so

$$1 + z = \nu_{obs}/\nu_{em} = (u_\alpha p^\alpha)_{obs}/(u_\alpha p^\alpha)_{em}$$

4-velocity for the orbiting satellite is $u^\alpha = (c\dot{t}, \dot{r}, t\dot{h}eta, \dot{\phi}) = (c\dot{t}, 0, 0, \dot{\phi})$

we know that on circular orbits we have $c^2\dot{t}^2 m = r^3 p\dot{h}i^2$ and for particles we had $\dot{t}^2 = (1 - 3m/r)^{-1}$.

substitute back and get $\dot{\phi}^2 = mc^2(1 - 3m/r)^{-1}/r^3$ so

$u^0 = c\dot{t} = c/(1 - 3m/r)^{1/2}$ and $u^3 = \sqrt{mc^2/[r^3(1 - 3m/r)]}$

so, thats our 4-velocity. what about the emitted light? lets make it simple and look only at the tangent points. here the light at teh moment of emission has only tangential motion so $\dot{r} = 0$ so $0 = (1 - 2m/r)c^2\dot{t}^2 - r^2\dot{\phi}^2$

but the standard E-L equation is $(1 - 2m/r)\dot{t} = E/c^2$ then

$0 = (1 - 2m/r)c^2E^2/[c^4(1 - 2m/r)^2] - r^2\dot{\phi}^2$

$0 = E^2/[c^2(1 - 2m/r)] - r^2\dot{\phi}^2$

$\dot{\phi}^2 = (E/c)^2/[r^2(1 - 2m/r)]$

$\dot{\phi} = \pm(E/c)/[r(1 - 2m/r)^{1/2}]$

4-momentum for the emitted light is $p^0 = c\dot{t}$ and $p^3 = \dot{\phi}$

$$p^\alpha = \Big(\frac{E/c}{(1 - 2m/r)}, 0, 0, \pm\frac{E/c}{r(1 - 2m/r)]^{1/2}}\Big)$$

now we do need the $\pm$ sign as on one side the light is emitted in the same direction as the particle is moving, and in the other its opposite.

for a stationary observer at infinity $u^\alpha = (c, 0, 0, 0)$, while they see the light with $p^\alpha = (E_\infty/c, 0, 0, 0)$. so $g_{\alpha\beta}u^\beta p^\alpha = E_\infty = h\nu_\infty$

but

$$(p^\alpha u_\alpha)_{em} = g_{\alpha\beta}p^\alpha u^\beta = g_{00}u^0p^0 + g_{33}u^3p^3$$

1st term

$$= (1 - 2m/r).(h\nu_\infty/c)/(1 - 2m/r).c/(1 - 3m/r)^{1/2} = (h\nu_\infty)/(1 - 3m/r)^{1/2}$$

2nd term

$$= \pm r^2\frac{(h\nu_\infty/c)}{r(1 - 2m/r)^{1/2}} \times \sqrt{\frac{mc^2}{r^3(1 - 3m/r)}}$$

$$= \pm h\nu_\infty \sqrt{\frac{m/r}{(1-2m/r)(1-3m/r)}}$$

$$h\nu_{em} = \frac{h\nu_\infty}{(1-3m/r)^{1/2}}[1 \pm \sqrt{\frac{m/r}{(1-2m/r)^{1/2}}}]$$

at r=6m $\nu_\infty/\nu_{em} = (1-.5)^{1/2}/[1+1/2] = 2\sqrt{1/2}/3 = 0.47$
and $= (1-.5)^{1/2}/[1-1/2] = 2\sqrt{1/2} = \sqrt{2} = 1.41$

so if this was instead a line emitted from the disc, we'd see it as broadened by all these effects!