5.13 Nature of the Event Horizon: acceleration (cont)

we had a satellite drop from rest at $r0$ rather than $r \to \infty$ and found that the E-L constant E is then

$$(1 - 2m/r0)(1 - 2m/r0)^{-1/2} = E/c^2$$

hence

$$(1 - 2m/r)\dot{t} = E/c^2 = (1 - 2m/r0)^{1/2}$$

so $\dot{t} = (1 - 2m/r0)^{1/2}/(1 - 2m/r)$

substitute this back into the metric

$$c^2 = (1 - 2m/r)c^2\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2$$

$$c^2(1 - 2m/r) = c^2(1 - 2m/r0) - \dot{r}^2$$

$$\dot{r}^2 = 2mc^2/r - 2mc^2/r0$$

so $2\dot{r}\ddot{r} = 2mc^2 d/d\tau(r^{-1}) = 2mc^2\dot{r}(-1/r^2)$ so $\ddot{r} = -mc^2/r^2$ this is for someone in the spacecraft and still everything is finite across the horizon

now lets see what a stationary observer at $r < r0$ measures. so we'll put it into coordinate time because coordinate time intervals are our grid points. we could do this in multiple ways, first we could go back to the metric and get dr/dt instead of $dr/d\tau$ - in which case we;d get

$c^2/\dot{t}^2 = (1 - 2m/r)c^2 - (1 - 2m/r)^{-1}dr/dt^2$

$$\frac{c^2(1 - 2m/r)^2}{1 - 2m/r0} = (1 - 2m/r)c^2 - (1 - 2m/r)^{-1}dr/dt^2$$

$$(dr/dt)^2 - = (1-2m/r)^2c^2 - \frac{c^2(1 - 2m/r)^3}{1 - 2m/r0} = (1-2m/r)^2c^2[(1 - \frac{1 - 2m/r}{1 - 2m/r0}]$$

$$= (1 - 2m/r)^2c^2\frac{2m/r - 2m/r0}{1 - 2m/r0}$$

so

$$dr/dt = \pm(1 - 2m/r)c\sqrt{\frac{2m/r - 2m/r0}{1 - 2m/r0}}$$

or we could have done it by chain rule

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = \frac{dr}{d\tau}\frac{1}{\dot{t}}$$

$$= \frac{\sqrt{2mc^2/r - 2mc^2/r0}(1-2m/r)c}{\sqrt{1-2m/r0}}$$

and got the same answer faster.

now the stationary observer measures time intervals $d\tau^2 = (1-2m/r)dt^2$ and proper distance $dR^2 = (1-2m/r)^{-1}dr^2$ so they measure a velocity

$$\Big(\frac{dR}{d\tau}\Big)^2 = \Big(\frac{dR}{dr}\frac{dr}{dt}\frac{dt}{d\tau}\Big)^2 = (1-2m/r)^{-1}\frac{(1-2m/r)^2}{1-2m/r0}(2mc^2/r - 2mc^2/r0)(1-2m/r)^{-1}$$

$$= \frac{(2mc^2/r - 2mc^2/r0)}{1-2m/r0}$$

so as $r \to 2m$ the stationary observer at r sees the spacecraft come past at c. even if the spacecraft was only dropped from an infinitesimal distance above r. so it accelerates from 0 to c in an infinitesimal distance $r0 - r$ so this means there are infinite accelerations at $r = 2m$ which means infinite forces.....

5.14 nature of the event horizon

Lets pull it all together. plainly $r = 2m$ is a coordinate singularity as $g_{rr} \to \infty$. but is there something more physical going on? we saw that at $r < 2m$ there is no such thing as fixed anything, so physically there is something happening. An observer at infinity watching an infalling particle - they see it initially accelerate towards the hole but as it gets closer then it slows down and appears to STOP at the horizon. Yet the infalling particle itself thinks it takes a finite time to get to $r = 0$ rather than running into any real barrier at $r = 2m$.

The way to think about this is to look at the scalar we can form from the Riemman curvature tensor $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$. This is NOT the same as the scalar we formed from contracting the Ricci tensor $R^\mu{}_\mu = g_{\mu\nu}R^{\mu\nu}$ - We saw that for empty space outside a gravitating body then this (and the Ricci tensor itself) were ZERO as the stress energy tensor and scalar are zero if space is

EMPTY and the Einstein equations can be written $R^{\mu\nu} = \kappa(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T)$. But the ricci tensor is a contraction of the full Riemann curvature and it is NOT necessarily true that all components of the Riemann curvature are zero if $R^{\mu\nu} = 0$ - there is real curvature of spacetime (there has to be as this is GRAVITY). This Riemann curvature scalar $\propto r^{-6}$. There IS a true curvature singularity, and its at $r = 0$ as expected.

If we draw the curvature of spacetime (rather than an embedding diagram which shows proper distance versus r) then we can see whats going on. Any freefalling observer is in an inertial frame - we can draw a flat bit of spacetime which is tangent to the spacetime curvature at that point. and since time is orthoganal to space then we can use this to define the t axis. and lightcones make a 45 degree angle as normal - its all locally inertial and fine and special relativity works, but its TILTED with respect to ∞ due to the spacetime curvature. at $r = 2m$ the tilt is 45 degrees, so light DOES NOT GET OUT. There is no way that lightcones can get out to infinity - they simply don't connect. so a black hole is simply spacetime so curved that not even light can get out.

5.15 Eddington -Finklestein coordinates

we can explore this by doing a coordinate transformation so that we get rid of the coordinate singularity at the horizon

for light, we know that $dr^2 = (1-2m/r)^2 c^2dt^2$ $c^2dt^2 = dr^2/(1-2m/r)^2 = dr_*^2$. Then radial null paths have constant $ct = \pm r_* + v$ where v is a constant for light, and the +ve is outgoing, -ve is ingoing. Integrate the radial bit and get $r_* = r + 2m \log|r/2m - 1|$.

Anything else has paths where v is not constant!! so we can change variable to $cdt = -r_* + v$ and then differentiate to get $cdt/dr = dr_*/dr + dv/dr$. hence $cdt = -dr/(1-2m/r) + dv$

$$c^2dt^2 = dv^2 - \frac{2dvdr}{(1-2m/r)} + \frac{dr^2}{(1-2m/r)^2}$$

replace the cdt in the metric with this and get

$$c^2d\tau^2 = (1-2m/r)\left[dv^2 - \frac{2dvdr}{(1-2m/r)} + \frac{dr^2}{(1-2m/r)^2}\right] - \frac{dr^2}{1-2m/r}$$

$= (1 - 2m/r)dv^2 - 2dvdr$

so light $c^2 d\tau^2 = 0$ is indeed $dv = 0$ - OR the other solution is $(1 - 2m/r)(dv/dr)^2 - 2(dv/dr) = 0$ i.e. $(1 - 2m/r)(dv/dr) = 2$ and $dv/dr = 2/(1 - 2m/r)$.

this changes sign at $r = 2m$. these are outgoing light paths, and they go to smaller radii for $r < 2m$. so even outgoing light rays go inwards!

But it makes more sense in our heads if light is diagonal, so we could tip the x axis so that the lines of constant v are diagonal. But an easier way is to go for $ct_* = v - r$. Hence $cdt_*/dr = dv/dr - 1$. So for ingoing its $cdt_*/dr = -1$ while outgoing are $cdt_*/dr = 2/(1-2m/r) - 1 = (2-(1-2m/r))/(1-2m/r) = (1 + 2m/r)/(1 - 2m/r)$ and again this changes sign at $r = 2m$ such that outgoing rays go inwards!