2.2 Coordinate transformations

Lets now think about more general spaces which have arbitrary curvature. Define a point P in some space, and another point a little further on called R. These points have NOTHING to do with the coordinate system. They exist irrespective of the labels we give them. Coordinates just make them easier to handle. So lets take two points, P and R, defined on coordinates x^a with basis vectors $\underline{e}_a$. Then these have position vectors $\underline{r}(P) = x^a(P)\underline{e}_a$ and $\underline{r}(Q) = x^a(Q)\underline{e}_a$ and the displacement vector which points from P to Q is $\underline{dr} = [x^a(Q) - x^a(P)]\underline{e}_a = dx^a\underline{e}_a$ ie whose components are the coordinate differences. A key property of tensors is that their representations in different coordinate systems depend only on the relative orientations and scales on the coordinate axes at that point and NOT on the absolute values of coordinates. $\underline{dr}$ will be the same in all coordinate systems, though its components will be different depending on our choice of coordinates.

lets go back to an arbitrary position vector

$$\underline{r} = x^1\underline{e_1} + x^2\underline{e_2} + ... = x^a\underline{e_a}$$

This has components x^a along whatever basis vectors $\underline{e_a} = \partial\underline{r}/\partial x^a$ we are using (they don't have to be rectilinear or orthoganal).

Now transform to a different coordinate system $x^{\overline{b}}$ - the vector is the SAME vector so $\underline{r} = x^a\underline{e_a} = x^{\overline{b}}\underline{e_{\overline{b}}}$

so again we get the basis vectors by the partial derivatives so $\underline{e_{\overline{b}}} = \partial\underline{r}/\partial x^{\overline{b}}$

and we can write the new (primed) coordinates, as function of the old (unprimed) ones so $x^{\overline{a}} = x^{\overline{a}}(x^b)$. (our conventions mean that this expands to N equations):

$$x^{\overline{1}} = x^{\overline{1}}(x^1, x^2....x^N)$$
$$x^{\overline{2}} = x^{\overline{2}}(x^1, x^2...x^N)$$

If this insead was about a fundtion $f = f(x^1, x^2...x^N)$ then we would instantly know how to do a total differential in terms of the partial differentials.

$$\Delta f = \frac{\partial f}{\partial x^1}\Delta x^1 + \frac{\partial f}{\partial x^2}\Delta x^2 + ...\frac{\partial f}{\partial x^N}\Delta x^N = \frac{\partial f}{\partial x^a}\Delta x^a$$

so we can write our coordinate transformations as (the N equations!)

$$\Delta x^{\overline{b}} = \frac{\partial x^{\overline{b}}}{\partial x^a}\Delta x^a$$

where there are N^2 separate $\partial x^{\overline{b}}/\partial x^a$. In the limit of infinitesimals this goes to

$$dx^{\overline{b}} = \frac{\partial x^{\overline{b}}}{\partial x^a}dx^a$$

A superscript index on the denominator counts the same in the summation convention as a subscript on the numerator.

2.3 Contravariant tensors of 1st order (4 vectors)

Entities which transform like the coordinate differences are called contravariant tensors of first order, or (1,0) tensors. They are defined by their transformation properties. If $\underline{A} = A^a \underline{e_a}$ has components which transform as

$$A^{\overline{b}} = \frac{\partial x^{\overline{b}}}{\partial x^a}A^a$$

then it is a contravariant tensor - things like 4-velocity, momentum, 4-force etc all transform like this.

2.4 What about the basis vectors?

Back to the position vector

$$\underline{r} = x^a \underline{e}_a = x^{\overline{b}}\underline{e_{\overline{b}}}$$

Old basis vectors are $\underline{e}_a = \partial\underline{r}/\partial x^a$. New ones are again the tangents to the coordinate curves so

$$e_{\overline{b}} = \frac{\partial\underline{r}}{\partial x^{\overline{b}}} = \frac{\partial\underline{r}}{\partial x^a}\frac{\partial x^a}{\partial x^{\overline{b}}} = \frac{\partial x^a}{\partial x^{\overline{b}}}\underline{e_a}$$

where the second step is simply the chain rule. This is a bit different. Our basis vectors don't transform like the coordinate differences i.e. like contravarient tensors which have $A^{\overline{b}} = \partial x^{\overline{b}}/\partial x^a A^a$ - they go the other way (derivative of the old with respect to the new, whereas the components of the vectors - contravarient tensors of first order - transformed as the derivatives of the new with respect to the old). So this looks more like a INVERSE transformation if we were talking about coordinates.

2.5 Example: basis vectors for spherical polar coordinates

we KNOW how to do coordinate transformations. eg for any point in a 3D space we can write

$$\underline{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where **i,j,k** are simply the basis vectors along the x,y,z axes. we can transform to spherical polar coordinates where $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. This is the old coordinates x,y,z i.e. x^i written as functions of the new i.e. r, θ, ϕ i.e. $x^{\bar{j}}$. then the new basis vectors along the new coordinate directions are

$$\underline{e}_{\bar{1}} = \underline{e}_r = \frac{\partial \underline{r}}{\partial r} = \sin\theta\cos\phi\ \mathbf{i} + \sin\theta\sin\phi\ \mathbf{j} + \cos\theta\ \mathbf{k}$$

$$\underline{e}_{\bar{2}} = \underline{e}_\theta = \frac{\partial \underline{r}}{\partial \theta} = r\cos\theta\cos\phi\ \mathbf{i} + r\cos\theta\sin\phi\ \mathbf{j} - r\sin\theta\mathbf{k}$$

$$\underline{e}_{\bar{3}} = \underline{e}_\phi = \frac{\partial \underline{r}}{\partial \phi} = -r\sin\theta\sin\phi\ \mathbf{i} + r\sin\theta\cos\phi\ \mathbf{j}$$

2.6 Covariant tensors of first order

Quantities which transform as the inverse of the coordinate differences are not just limited to the basis vectors. If we go back into 3D and look at how the components of a scalar field $\phi(x^i)$ transform. Scalar fields are just numbers at a given point. whatever coordinates you give the point makes no change in the number. But its gradient will change depending on coordinate system (because we take gradients as change in number over some coordiante distance). Its gradient is simply $\nabla\phi(x^i)$ which has components

$$\frac{\partial \phi}{\partial x^i}$$

so go to another frame with primed coordindates and look at $\nabla\phi(x^{\bar{j}})$ which has components

$$\frac{\partial \phi}{\partial x^{\bar{j}}} = \frac{\partial \phi}{\partial x^i}\frac{\partial x^i}{\partial x^{\bar{j}}}$$

so the gradient is NOT a contravariant tensor. So there are physical quantities we will need to describe which are NOT contravarient. Gradients have components which transform the other way to the way a contravariant tensor transforms. Goes as old wrt new not new wrt old i.e. like the normal basis vectors. We call things that transform like this COVARIANT vectors (also called (0,1) tensors or covariant tensors of the first kind/order, or one forms) and we denote these by a lower index on the components (like the normal basis vectors are denoted by a lower index), and transformation laws for the components (now in generalised N dimensions) are:

$$A_{\overline{b}} = \frac{\partial x^a}{\partial x^{\overline{b}}} A_a$$

2.7 Higher order tensors

Now lets think about other things we can do. What is $T^{ab} = A^a B^b$?

$$T^{\overline{cd}} = A^{\overline{c}} B^{\overline{d}} = \frac{\partial x^{\overline{c}}}{\partial x^e} A^e \frac{\partial x^{\overline{d}}}{\partial x^f} B^f = \frac{\partial x^{\overline{c}}}{\partial x^e} \frac{\partial x^{\overline{d}}}{\partial x^f} A^e B^f = \frac{\partial x^{\overline{c}}}{\partial x^e} \frac{\partial x^{\overline{d}}}{\partial x^f} T^{ef}$$

This transforms like 2 powers of the coordinate differentials so we call it a second order contravariant tensor or (2,0). Getting higher order tensors from lower order ones in this way is called an outer product.

2.8 Tensor algebra

$T^a = \kappa(A^a + B^a)$. what is this? always write it in another frame and see how it transforms

$$\begin{aligned} T^{\overline{b}} &= \kappa(A^{\overline{b}} + B^{\overline{b}}) \\ &= \kappa\Big(\frac{\partial x^{\overline{b}}}{\partial x^a} A^a + \frac{\partial x^{\overline{b}}}{\partial x^a} B^a\Big) \\ &= \frac{\partial x^{\overline{b}}}{\partial x^a} T^a \end{aligned}$$

so tensors are linear (we can add them, multiply by constants and it leaves their nature unchanged - they are still tensors of the same order)

2.9 Metric tensor - 2nd order covariant tensor

Lets go back to our generalised differential position vector again $d\underline{r} = dx^a \underline{e}_a$ This is telling us the distance between the two points. We can get the length of this vector via the dot product

$$ds^2 = d\underline{r}.d\underline{r} = dx^a \underline{e}_a.dx^b \underline{e}_b$$

$$= \underline{e}_a.\underline{e}_b dx^a dx^b$$

$$= g_{ab} dx^a dx^b$$

where $g_{ab} = \underline{e}_a.\underline{e}_b$ is called the metric. Its transformation under coordinate change can be seen as we derived the basis vector transformations

$$\underline{e}_{\overline{a}}.\underline{e}_{\overline{b}} = \frac{\partial x^c}{\partial x^{\overline{a}}} \underline{e}_c . \frac{\partial x^d}{\partial x^{\overline{b}}} \underline{e}_d$$

$$= \frac{\partial x^c}{\partial x^{\overline{a}}} \frac{\partial x^d}{\partial x^{\overline{b}}} \underline{e}_c.\underline{e}_d$$

So the components transform like the basis vectors twice - called covariant tensor of second order - this is the METRIC tensor and its wildly important in everything that we are going to do next, because it includes what happens to distances (spacetime intervals!!) if the space is curved.

From its definition in terms of the dot product of basis vectors then we can see that $g_{ab} = \underline{e}_a.\underline{e}_b = \underline{e}_b.\underline{e}_a = g_{ba}$, ie that it is symmetric. It is basically a rule for taking two vectors and getting a single number out.