2.11 Example metric for 3D flat space

lets do an example to make this all less abstract! The position vector between two points close together in flat 3D space is simply $d\underline{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$. So the distance between these two points is $ds^2 = d\underline{r}.d\underline{r} = dx^2 + dy^2 + dz^2$ which is exactly what we expect from Pythagoros!

and we can do this our new was as well as the dot product of basis vectors. these are $\underline{e}_1 = \mathbf{i}$, $\underline{e}_2 = \mathbf{j}$, $\underline{e}_3 = \mathbf{k}$ where $x^1 = x$, $x^2 = y$ and $x^3 = z$. then $g_{11} = g_{xx} = \mathbf{i}.\mathbf{i} = 1$, $g_{22} = g_{yy} = \mathbf{j}.\mathbf{j} = 1$ and $g_{33} = g_{zz} = \mathbf{k}.\mathbf{k} = 1$, and all cross terms are zero. so we get our distance by expanding our double sum $ds^2 = (g_{ij}dx^i)dx^j$ explicitally as

$$= (g_{1j}dx^1 + g_{2j}dx^2 + g_{3j}dx^3)dx^j$$

$$= g_{1j}dx^1dx^j + g_{2j}dx^2dx^j + g_{3j}dx^3dx^j$$

$$= g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + g_{13}dx^1dx^3 +$$

$$g_{21}dx^2dx^1 + g_{22}dx^2dx^2 + g_{23}dx^2dx^3 +$$

$$g_{31}dx^1dx^1 + g_{32}dx^1dx^2 + g_{33}dx^3dx^3$$

all the $g_{ij} = 0$ except where $i = j$ for this coordinate system so

$$= g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3 = dx^2 + dy^2 + dz^2$$

but we also looked at how we could do a coordinate transformation to spherical polar coordinates, where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. last lecture we saw that this gave us a new set of basis vectors, $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{e}_\phi$ which we could define in terms of our old basis vectors **i,j,k** as

$$\underline{e}_r = \frac{\partial\underline{r}}{\partial r} = \sin\theta\cos\phi\ \mathbf{i} + \sin\theta\sin\phi\ \mathbf{j} + \cos\theta\ \mathbf{k}$$

$$\underline{e}_\theta = \frac{\partial\underline{r}}{\partial\theta} = r\cos\theta\cos\phi\ \mathbf{i} + r\cos\theta\sin\phi\ \mathbf{j} - r\sin\theta\mathbf{k}$$

$$\underline{e}_\phi = \frac{\partial\underline{r}}{\partial\phi} = -r\sin\theta\sin\phi\ \mathbf{i} + r\sin\theta\cos\phi\ \mathbf{j}$$

from above we can calculate how distance depends on these new coordinates by taking the dot products

$$g_{rr} = \underline{e}_r.\underline{e}_r = \sin\theta^2\cos^2\phi + \sin\theta^2\sin^2\phi + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1$$

$$g_{\theta\theta} = \underline{e}_\theta . \underline{e}_\theta = r^2 \cos^2\theta \cos^2\phi + r^2 \cos^2\theta \sin^2\phi + (-r)^2 \sin^2\theta = r^2 \cos^2\theta + r^2 \sin^2\theta = r^2$$

$$g_{\phi\phi} = (-r)^2 \sin^2\theta \sin^2\phi + r^2 \sin^2\theta \cos^2\phi = r^2 \sin^2\theta$$

there are also the cross terms $g_{r\theta} = g_{\theta r}$ etc. but these are all zero. so then our utterly general (and completely opaque!) distance is $ds^2 = g_{ij} dx^i dx^j$. Expanding out this double sum explicitally gives

$$ds^2 = (g_{ij} dx^i) dx^j = (g_{1j} dx^1 + g_{2j} dx^2 + g_{3j} dx^3) dx^j$$

$$= g_{1j} dx^1 dx^j + g_{2j} dx^2 dx^j + g_{3j} dx^3 dx^j$$

$$= g_{11} dx^1 dx^1 + g_{12} dx^1 dx^2 + g_{13} dx^1 dx^3 +$$

$$g_{21} dx^2 dx^1 + g_{22} dx^2 dx^2 + g_{23} dx^2 dx^3 +$$

$$g_{31} dx^1 dx^1 + g_{32} dx^1 dx^2 + g_{33} dx^3 dx^3$$

this coordinate system is $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$ and in in this system all the $g_{ij} = 0$ if $i \neq j$. so we can throw out most of these terms to get

$$ds^2 = g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

which is instantly recognissable as the distance between two points in 3D flat space in spherical polar coordinates. So this abstract mathematical machinary really does connect to what we already know!

Curvature is completely defined by the metric tensor! its the property of the space, how distance relates to position. BUT, we still have a way to go as this is NOT the sort of way we want to define curvature. it contains the important information about the real curvature BUT it also contains extraneous information about the coordinate system we are using. Our first cartesian set of coordinates obviously gave us flat space but the second is more subtle. at first glance you might think that it represents a curved surface as distances are no longer doing a Pythagoros law - the metric tensor components are not given by a diagonal, unit matrix. BUT - there exists a coordinate transformation that gets us back to a metric in the form of $ds^2 = dx^2 + dy^2 + dz^2$ - the underlying space is FLAT.

The difference in a real curved space is that there is no transformation we can make to get this metric back to a flat space one. But its not necessarily immediately apparent from the components of the metric tensor which ones will allow coordinate transformations to get us to the unit matrix.

2.12 Kronekar delta and invariance of tensor equations

we saw that the basis vectors transform as $\underline{e}_{\overline{b}} = \partial x^a/\partial x^{\overline{b}}\underline{e}_a$. This means that any quantity $\underline{A} = A^a\underline{e}_a$ in another frame,

$$A^{\overline{b}}\underline{e}_{\overline{b}} = \frac{\partial x^{\overline{b}}}{\partial x^a}A^a\frac{\partial x^d}{\partial x^{\overline{b}}}\underline{e}_d$$

where we changed a to d in the implied sum for the basis vectors as we've already used a - we can call them anything we like in a sum, but they are BOTH sums. So finally we get

$$\frac{\partial x^d}{\partial x^a}A^a\underline{e}_d = \delta^d_a A^a\underline{e}_d = A^a\underline{e}_a$$

where the $\delta^a_b = 1$ for $a = b$ and 0 otherwise is termed the Kronekar delta function. So if we have something that transforms as the coordinate differences, then this means its tensor equation looks the same in ANY frame!!!

A quick way to see if you have dropped any indices is to count them - cancel out the indices which have an impled sum, and then the indices we have on the left hand side MUST be equal to the indices we have on the right.

2.13 Basis vectors for covariant components

Covariant components came from $\nabla\phi$ - but this in cartesian coordinates is just

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = A_i\underline{e}^i$$

where $A_i = \partial\phi/\partial x^i$ are the components, and since we know that these are covariant, the basis vectors must have the index high.

and we would get these basis vectors in a different way - $\underline{e}^1 = \mathbf{i} = \nabla x^1$, $\underline{e}^2 = \mathbf{j} = \nabla x^2$ and $\underline{e}^3 = \mathbf{k} = \nabla x^3$

so the basis vectors are the same as before, but we'd get them in a different way. $e^i = \nabla x^i$ which is not necessarily equal to $e_i = \partial\underline{r}/\partial x^i$.

show what happens if we go to a new coordinate frame $x^{\overline{j}}$

$$\nabla\phi = \frac{\partial\phi}{\partial x^i}\underline{e}^i$$

$$= \frac{\partial \phi}{\partial x^{\bar{j}}} \frac{\partial x^{\bar{j}}}{\partial x^i} \underline{e}^i$$

$$= \frac{\partial \phi}{\partial x^{\bar{j}}} \nabla x^{\bar{j}}$$

so this tells us how these new basis vectors transform

$$e^{\bar{j}} = \nabla x^{\bar{j}} = \frac{\partial x^{\bar{j}}}{\partial x^i} \underline{e}^i$$

and then $\underline{A} = A_a \underline{e}^a$ will transform to another frame as

$$A_{\bar{b}} \underline{e}^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a} A_a \frac{\partial x^{\bar{b}}}{\partial x^c} \underline{e}^c = \frac{\partial x^c}{\partial x^a} A_a \underline{e}^c = \delta^c_a A_a \underline{e}^c = A_a \underline{e}^a$$

So our tensor equations look the same in all frames!

FOR MATHEMATICALLY INCLINED PEOPLE ONLY!! This gives us another vector space of the manifold - call it T_P^* the dual or cotangent space of the manifold at P as opposed to our first set which defined the tangent space T_P to the manifold at P

Back to everyone. The basis vectors themselves look identical if we have an orthonormal set of coordinates (they don't have to be reclilinear, just be 90 degrees where they meet). BUT THEY ARE NOT IDENTICAL IF THE COORDINATES ARE NOT ORTHOGANAL.

e.g. in 3D spherical polars $r^2 = x^2 + y^2 + z^2$ so

$$\underline{e}^r = \nabla r = \frac{\partial r}{\partial x} \mathbf{i} + \frac{\partial r}{\partial x} \mathbf{j} + \frac{\partial r}{\partial z} \mathbf{k}$$

easy way is to say $\partial r^2 / \partial x = 2r \partial r / \partial x$ so $\partial r / \partial x = 1/(2r) \times \partial r^2 / \partial x = 1/(2r) \times \partial (x^2 + y^2 + z^2) / \partial x = 2x/(2r) = x/r$

similarly $\partial r / \partial y = y/r$ and $\partial r / \partial z = z/r$

so $\underline{e}^r = (x/r)\mathbf{i} + (y/r)\mathbf{j} + (z/r)\mathbf{k}$. but we know $x = r \sin\theta \cos\phi$ etc so

$\underline{e}^r = \sin\theta \cos\phi \mathbf{i} + \sin\theta \sin\phi \mathbf{j} + \cos\theta \mathbf{k}$

etc for the rest.