2.14 Basis vectors for covariant components - 2

Covariant components came from $\nabla\phi$ - but this in cartesian coordinates is just

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

so these LOOK like they have the same basis vectors as standard, where $e_j = \partial \underline{r}/\partial x^j$. BUT THEY DON'T!

we can see this by transforming to another, arbitrary frame with coordinates $x^{\bar{i}} = x^{\bar{1}}, x^{\bar{2}}, x^{\bar{3}}$ then we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x^{\bar{1}}}\frac{\partial x^{\bar{1}}}{\partial x} + \frac{\partial\phi}{\partial x^{\bar{2}}}\frac{\partial x^{\bar{2}}}{\partial x} + \frac{\partial\phi}{\partial x^{\bar{3}}}\frac{\partial x^{\bar{3}}}{\partial x}$$

we can do the same for $\partial\phi/\partial y$ and $\partial\phi/\partial z$. And then we get

$$\nabla\phi = \frac{\partial\phi}{\partial x^{\bar{1}}}\nabla x^{\bar{1}} + \frac{\partial\phi}{\partial x^{\bar{2}}}\nabla x^{\bar{2}} + \frac{\partial\phi}{\partial x^{\bar{3}}}\nabla x^{\bar{3}}$$

so this makes it clear that the basis vectors we want are $\nabla x^{\bar{i}}$ for our transformed frame

$$= \frac{\partial\phi}{\partial x^{\bar{1}}}\underline{e}^{\bar{1}} + \frac{\partial\phi}{\partial x^{\bar{2}}}\underline{e}^{\bar{2}} + \frac{\partial\phi}{\partial x^{\bar{3}}}\underline{e}^{\bar{3}}$$

$$= \frac{\partial\phi}{\partial x^{\bar{i}}}\underline{e}^{\bar{i}}$$

so we'd better have the basis vectors in the original frame be ∇x^i as well - which is the SAME as **i,j,k** for the special case of cartesian coordinates, where the axes are orthoganal and rectilinear. BUT NOT ALL COORDINATE FRAMES ARE LIKE THIS, especially when we get to curvature.

so lets look explicitally at how these transform

$$\underline{e}^{\bar{b}} = \nabla x^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}\nabla x^a = \frac{\partial x^{\bar{b}}}{\partial x^a}\underline{e}^a$$

and then $\underline{A} = A_a\underline{e}^a$ will transform to another frame as

$$A_{\bar{b}}\underline{e}^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}A_a\frac{\partial x^{\bar{b}}}{\partial x^c}\underline{e}^c = \frac{\partial x^c}{\partial x^a}A_a\underline{e}^c = \delta^c_a A_a\underline{e}^c = A_a\underline{e}^a$$

So our tensor equations look the same in all frames!

FOR MATHEMATICALLY INCLINED PEOPLE ONLY!! This gives us another vector space of the manifold - call it T_P^* the dual or cotangent space of the manifold at P as opposed to our first set which defined the tangent space T_P to the manifold at P

Back to everyone. The basis vectors themselves look identical if we have an orthonormal set of coordinates (they don't have to be reclilinear, just be 90 degrees where they meet). BUT THEY ARE NOT IDENTICAL IF THE COORDINATES ARE NOT ORTHOGANAL.

2.15 Covariant and contravariant: more on the metric

But if we have another set of basis vectors IN OUR UNPRIMED FRAME then we can write any arbitrary vector either on the old basis in the tangent space OR the new basis in the cotangent space i.e. $\underline{\lambda} = \lambda^a \underline{e}_a = \lambda_b \underline{e}^b$. If the basis vectors are the same i.e. we had orthonormal bases then the contravariant and covariant components are IDENTICAL. But of course in general they are not.

For the pure mathematically inclined, setting the contravariant and covariant forms of writing this as equal is a bit of a fudge as $\lambda^a \underline{e}_a$ and $\lambda_b \underline{e}^b$ are actually are in two different vector spaces T_P and $T*_P$, respectively. But we'll do a bit more on this later

Take another vector $\underline{\mu} = \mu^a \underline{e}_a = \mu_a \underline{e}^a$. The dot product of these is

$$\underline{\lambda}.\underline{\mu} = \lambda^a \underline{e}_a.\mu^b \underline{e}_b = \lambda^a \mu^b \underline{e}_a.\underline{e}_b = \lambda^a \mu^b g_{ab} = |\underline{A}||\underline{B}| \cos\chi$$

where χ is the angle between the two vectors. But we could have done this with the covariant components alone

$$\underline{\lambda}.\underline{\mu} = \lambda_a \underline{e}^a.\mu_b \underline{e}^b = \lambda_a \mu_b \underline{e}^a.\underline{e}^b = \lambda_a \mu_b g^{ab}$$

where we have defined $g^{ab} = \underline{e}^a.\underline{e}^b$. Again this is symmetric by definition of the dot product so $g^{ab} = g^{ba}$. But we could also have done this with the mixed components

$$\underline{\lambda}.\underline{\mu} = \lambda^a \underline{e}_a.\mu_b \underline{e}^b = \lambda^a \mu_b \underline{e}_a.\underline{e}^b$$

What is this dot product ? lets do it explicitally in 3D

$$\underline{e}_i.\underline{e}^j = \frac{\partial \underline{r}}{\partial x^i}.\nabla x^j$$

where $\underline{r} = x^i \underline{e}_i$. Then we have

$$e_i.e^j = (\frac{\partial x^1}{\partial x^i} i + \frac{\partial x^2}{\partial x^i} j + \frac{\partial x^3}{\partial x^i} k).(\frac{\partial x^j}{\partial x^1} i + \frac{\partial x^j}{\partial x^2} j + \frac{\partial x^j}{\partial x^3} k)$$
$$= \frac{\partial x^1}{\partial x^i}\frac{\partial x^j}{\partial x^1} + \frac{\partial x^2}{\partial x^i}\frac{\partial x^j}{\partial x^2} + \frac{\partial x^3}{\partial x^i}\frac{\partial x^j}{\partial x^3}$$
$$= \frac{\partial x^j}{\partial x^i} = \delta_i^j$$

where the kronekar delta function is $\delta_i^j = 1$ for $i = j$, 0 otherwise. NB because of the summation convention then $\delta_i^i = \delta_1^1 + \delta_2^2 + \delta_3^3 = 3$ So now we have

$$\lambda^a \underline{e}_a.\mu_b \underline{e}^b = \lambda^a \mu_b \underline{e}_a.\underline{e}^b = \lambda^a \mu_b \delta_a^b = \lambda^a \mu_a$$

By symmetry of the dot product and metric then $\lambda^a \mu_a = \lambda_a \mu^a$. So collecting all these together we have

$$\lambda^a \mu_a = \lambda_a \mu^a = g_{ab} \lambda^a \mu^b = g^{ab} \lambda_a \mu_b$$

So this gives us an easy way of swapping between contravariant and covariant components - we use the METRIC $g_{ab} \lambda^a \mu^b = \lambda_a \mu^a$ so

$$\lambda_a = g_{ab} \lambda^b$$

and as $g^{ab} \lambda_a \mu_b = \lambda^b \mu_b$ so

$$\lambda^a = g^{ab} \lambda_b$$

Tensors derived from other tensors by raising or lowering the indices via the metric are called **associated tensors**.

We also get another useful metric relation in that $\lambda_a = g_{ab} \lambda^b = g_{ab} g^{bc} \lambda_c$ so

$$g_{ab} g^{bc} = \delta_c^a$$

For the pure mathematically inclined, heres how its done. A contravariant vector $\underline{\lambda} = \lambda^a \underline{e}_a$ in T_p is associated with a covariant vector $\tilde{\lambda} = \lambda_a \underline{e}^a$ in $T*_P$ via the metric (or dot or inner product) such that for all vectors $\underline{\mu}$ in T_P then $\tilde{\lambda}(\underline{\mu})$ is a real number $= \mu^a \lambda_a$. i.e. it define covariant vectors as linear functions which map covariant vectors (one–forms) into real numbers via the metric. Schutz takes this approach (eventualy), as do Misner, Thorne and Wheeler. Foster and Nightingale take the approach I've used here.

2.16 Summary: Tensor definitions and algebra

Suppose that at a point P on a manifold M there are N^{r+s} quantities $\tau^{a_1...a_r}{}_{b_1...b_s}$ which under a change of coordinates transform as

$$\tau^{\overline{a_1}...\overline{a_r}}{}_{\overline{b_1}...\overline{b_s}} = \frac{\partial x^{\overline{a_1}}}{\partial x^{c_1}}...\frac{\partial x^{\overline{a_r}}}{\partial x^{c_r}}\frac{\partial x^{d_1}}{\partial x^{\overline{b_1}}}...\frac{\partial x^{d_s}}{\partial x^{\overline{b_s}}}\tau^{c_1...c_r}{}_{d_1...d_s}$$

where the partial derivatives of the coordinates are also evaluated at P. Then the quantities $\tau^{a_1...a_r}{}_{b_1...b_s}$ are called components of as type (r,s) tensor at P. The sum (r+s) is called the rank or order of the tensor, (r,0) might be called a contravariant tensor of rank r, while a (0,s) might be called a covariant tensor of rank s. If both r and s are nonzero then its called a mixed tensor. scalars are type (0,0) tensors. The whole point is that tensor equations maintain physical meaning under coordinate transformation.

linear combination

$$T^c_d = aA^c_d + bB^c_d$$

where a and b are scalars and A^c_d, B^c_d are (1,1) mixed tensors. Then T^c_d is also a (1,1) mixed tensor.

$$T^{\overline{e}}_{\overline{f}} = aA^{\overline{e}}_{\overline{f}} + bB^{\overline{e}}_{\overline{f}} = a\frac{\partial x^{\overline{e}}}{\partial x^c}\frac{\partial x^d}{\partial x^{\overline{f}}}A^c_d + b\frac{\partial x^{\overline{e}}}{\partial x^c}\frac{\partial x^d}{\partial x^{\overline{f}}}B^c_d$$

$$= \frac{\partial x^{\overline{e}}}{\partial x^c}\frac{\partial x^d}{\partial x^{\overline{f}}}T^c_d$$

Direct (outer) Products We can form higher order tensors out of lower order ones by multiplying the components.

$$T^a_b = A^a B_b$$

Again, its easy to show that this transforms as required.

$$T^{\overline{a}}_{\overline{b}} = A^{\overline{a}} B_{\overline{b}} = \frac{\partial x^{\overline{a}}}{\partial x^c}A^c\frac{\partial x^d}{\partial x^{\overline{b}}}B_d = \frac{\partial x^{\overline{a}}}{\partial x^c}\frac{\partial x^d}{\partial x^{\overline{b}}}T^c_d$$

General Inner product (contraction)

Set an upper and lower index as equal i.e. summing it, gives a new tensor of rank 2 less than the original one. Generally we keep the same kernal letter e.g. if we have a tensor $T^{a\ \ cd}_{\ b}$ then $T^{ac} = T^{a\ \ cd}_{\ d}$. Though obviously this is not unique as we could have also had the DIFFERENT contractions $T^{ad} = T^{a\ \ bd}_{\ b}$ or $T^{cd} = T^{a\ \ cd}_{\ a}$. Lets check that this is a valid tensor operation, and see how it transforms:

$$T^{\overline{a}\ \ \overline{cd}}_{\ \overline{a}} = \frac{\partial x^{\overline{a}}}{\partial x^e}\frac{\partial x^f}{\partial x^{\overline{a}}}\frac{\partial x^{\overline{c}}}{\partial x^g}\frac{\partial x^{\overline{d}}}{\partial x^h}T^{e\ \ gh}_{\ f} = \delta_e^{\ f}\frac{\partial x^{\overline{c}}}{\partial x^g}\frac{\partial x^{\overline{d}}}{\partial x^h}T^{e\ \ gh}_{\ f} = \frac{\partial x^{\overline{c}}}{\partial x^g}\frac{\partial x^{\overline{d}}}{\partial x^h}T^{e\ \ gh}_{\ e} = \frac{\partial x^{\overline{c}}}{\partial x^g}\frac{\partial x^{\overline{d}}}{\partial x^h}T^{gh}$$

Raising and lowering indexes

This is simply a direct product of a tensor with the metric, followed by a contraction. We can use it to swap between covariant and contravariant components AND basis vectors so $A_a = g_{ab}A^b$, $\underline{e}^a = g^{ab}\underline{e}_b$ where $g^{ab}g_{bc} = \delta^a_c$

Inner Product of 2 vectors (dot or scalar product)

$$\underline{A}.\underline{B} = g_{ab}A^aB^b = A^aB_a = A_bB^b = A^1B_1 + A^2B_2 + ...A^NB_N = |\underline{A}||\underline{B}|\cos\chi$$

where χ is the angle between the two vectors. Because its a scalar it must be invarient. We can see this explicitly on a change of coordinates

$$A^{\overline{a}}B_{\overline{a}} = \frac{\partial x^{\overline{a}}}{\partial x^c}A^c\frac{\partial x^d}{\partial x^{\overline{a}}}B_d$$

$$= \delta^d_cA^cB_d = A^cB_c$$

An obvious and VERY important invarient scalar product is $ds^2 = c^2d\tau^2 = dx^\alpha dx_\alpha = g_{\alpha\beta}dx^\alpha dx^\beta$

ANY EQUATION WILL BE INVARIANT UNDER GENERAL COORDINATE TRANSFORMATIONS IF IT STATES THE EQUALITY OF TWO TENSORS WITH THE SAME UPPER AND LOWER INDICES. $A^a_{\ bc} = B^a_{\ bc}$ can be written in another coordinate frame as $A^{\overline{a}}_{\ \overline{bc}} = B^{\overline{a}}_{\ \overline{bc}}$ since both sides transform between coordinate frames in the same way