3 Curved Space

We're going to assume that we'll always be dealing with a space which is continuous and differentiable - a differentiable manifold. We're also going to assume that it has a metric. If the metric is positive definite (inner prodscts +ve) then this is called a Riemannian manifold. Our metric won't be - so called pseudo–Riemannian. We've seen that the metric can carry (unimportant) information about coordinates - if space is flat then there is always some coordinate choice which gets it back into the unit matrix, and that the metric can also carry important non–transformable away information about the space curvature e.g. the metric of SR spacetime and the metric's we'll be using to describe the curved spacetime of gravity.

So the whole idea is that there is no 'force-at-a-distance' gravity. Paths curve only because they are following locally curved spacetime (which is curved because of mass). Any paths which follow the local curvature of spacetime (free-fall) are inertial frames, and we know how to do physics in inertial frame so we need to find out what these paths are.

3.1 Parallel transport

Some strange things happen when space is curved rather than flat. In flat space, we can take a vector from point and it keeps its direction - this is called parallel transport. This is important as when we do differentials we're comparing a vector at some point with a vector at another point. In curved space e.g. the surface of a sphere it just doesn't happen and the direction the vector points at the end of the path depends on the PATH as well as the start and end points. This is a really big problem when we start to look at differentiation.

Lets think some more about this, and have our vector $\underline{\lambda} = \lambda^a(s)\underline{e}_a$ defined along some curve given by coordinates $x^a(s)$. Suppose we tried to define derivitive in the 'obvious' way as

$$\frac{d\lambda^a}{ds} = \frac{\lim}{\delta s \to 0} \frac{[\lambda^a(s+\delta s) - \lambda^a(s)]}{\delta s}$$

For a start this isn't a tensor as it doesn't transform.

$$\frac{d\lambda^{\overline{a}}}{ds} = \frac{d(\partial x^{\overline{a}}/\partial x^b \lambda^b)}{ds} = \frac{\partial x^{\overline{a}}}{\partial x^b}\frac{d\lambda^b}{ds} + \lambda^b \frac{d}{ds}\frac{\partial x^{\overline{a}}}{\partial x^b} = \frac{\partial x^{\overline{a}}}{\partial x^b}\frac{d\lambda^b}{ds} + \frac{\partial^2 x^{\overline{a}}}{\partial x^b \partial x^c}\frac{dx^c}{ds}\lambda^b$$

Theres a funny extra term, which comes from the fact that $\lambda^a(u+\delta s)$ and $\lambda^a(s)$ are located at different points. In general the coordiate transforms depend on position so $\partial x^{\overline{a}}/\partial x^b$ evaluated at $u+\delta s$ is not the same as $\partial x^{\overline{a}}/\partial x^b$ at u. For differentiation to give a vector (tensor) we must take component differences at the same point. In flat space we slide one of the vectors to the other one by moving one of the vectors parallel to itself (called parallel transport).

So how do we parallelly transport a vector ? We don't change its length or direction so

$$\frac{d\underline{\lambda}}{ds} = 0 = \frac{d\lambda^a \underline{e_a}}{ds} = \frac{d\lambda^a}{ds}\underline{e_a} + \lambda^a \frac{d\underline{e_a}}{ds} = \frac{d\lambda^a}{ds}\underline{e_a} + \lambda^a \frac{\partial \underline{e_a}}{\partial x^b}\frac{dx^b}{ds}$$

There is this nasty component $\partial \underline{e_a}/\partial x^b$ but since this is the derivitive of a vector then it is itself a vector so we can write it as a linear combination of basis vectors

$$\frac{\partial \underline{e_a}}{\partial x^b} = \Gamma^c{}_{ab}\underline{e_c}$$

these are called Christoffel symbols or connection coefficients. We have written them in tensor notation but THEY DO NOT TRANSFORM AS TENSORS IN GENERAL.

$$= \frac{d\lambda^a}{ds}\underline{e_a} + \lambda^a \Gamma^c{}_{ab}\frac{dx^b}{ds}\underline{e_c} = \frac{d\lambda^a}{ds}\underline{e_a} + \lambda^b \Gamma^a{}_{bc}\frac{dx^c}{ds}\underline{e_a}$$

so since this is all the same vector component we can see that

$$\frac{d\lambda^a}{ds} + \lambda^b \Gamma^a{}_{bc}\frac{dx^c}{ds} = 0$$

So if we parallelly transport a vector λ at u to $u+\delta u$ we have

$$\frac{\delta\lambda^a}{\delta s} + \lambda^b \Gamma^a{}_{bc}\frac{\delta x^c}{\delta s} = 0$$

$$\delta\lambda^a = -\lambda^b \Gamma^a{}_{bc}\frac{\delta x^c}{\delta s}\delta s$$

$$\delta\lambda^a = -\lambda^b \Gamma^a{}_{bc}\delta x^c$$

so the parallely transported vector has components $\lambda^a_{||}(u+\delta s) = \lambda^a(s) - \lambda^b \Gamma^a{}_{bc}\delta x^c$, and we can compare this with the vector $\underline{\lambda}(u+\delta s)$ and use this to define absolute derivitive. The point is that $\underline{\lambda}(u+\delta s)$ contains two differences with respect to $\underline{\lambda}(s)$ - firstly there can be real physical differences - maybe this contravariant tensor is telling us about velocity and maybe there has been some real physics going on - accelerations and decelerations, that mean that $\underline{\lambda}(u+\delta s)$ is different from $\underline{\lambda}(s)$. BUT THERE CAN ALSO BE CHANGES BETWEEN THESE TWO VECTORS JUST BECAUSE THE SPACE IS CURVED. And we want to take out these 'space changes' which are what the parallelly transported vector tells us about, so that we can see the real physical changes. So these are

$$\frac{\lambda^a(u+\delta s) - \lambda^a_{||}(u+\delta s)}{\delta s} = \frac{\lambda^a(s) + \frac{d\lambda^a}{ds}\delta s - \lambda^a(s) + \lambda^b \Gamma^a{}_{bc}\delta x^c}{\delta s} = \frac{d\lambda^a}{ds} + \lambda^b \Gamma^a{}_{bc}\frac{\delta x^c}{\delta u}$$

so

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \lambda^b \Gamma^a{}_{bc}\frac{dx^c}{ds}$$

We saw that $d\lambda^a/ds$ is not a tensor, and I've told you that $\Gamma^a{}_{bc}$ isn't one either. But the two non-tensor bits concel each other out so $D\lambda^a/ds$ DOES transform as a tensor.

So this is the way to define physical or absolute derivative - use parallel transport to take out ths SPACE changes. But what we are left with is PATH DEPENDANT in general.

3.2 Covariant derivative

We can do this in terms of coordinates as opposed to a single parameter, and remove that path dependance. We had

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \lambda^b \Gamma^a{}_{bc}\frac{dx^c}{ds} = \frac{\partial\lambda^a}{\partial x^c}\frac{dx^c}{ds} + \lambda^b \Gamma^a{}_{bc}\frac{dx^c}{ds} = \Big(\frac{\partial\lambda^a}{\partial x^c} + \lambda^b \Gamma^a{}_{bc}\Big)\frac{dx^c}{ds}$$

So the components of the covariant derivative, are

$$\lambda^a{}_{;c} = \frac{\partial\lambda^a}{\partial x^c} + \Gamma^a{}_{bc}\lambda^b$$

This is obviously closely related to the absolute derivative, which is a (0,1) tensor, so this is a (1,1) tensor

3.3 Christoffel symbols and the metric

What are these horrors ? well they come from the derivitives of the tangent basis vectors, but these themselves are defined as $\underline{e_a} = \partial \underline{r} / \partial x^a$ so

$$\Gamma^c{}_{ab}\underline{e_c} = \frac{\partial \underline{e_a}}{\partial x^b} = \frac{\partial^2 \underline{r}}{\partial x^a \partial x^b} = \frac{\partial^2 \underline{r}}{\partial x^b \partial x^a} = \frac{\partial \underline{e_b}}{\partial x^a} = \Gamma^c{}_{ba}\underline{e_c}$$

so they are symmetric (changing the order of integration assumes that the second partial derivitives are continuous). To get some more about them the next step is to look at how the metric changes as a function of the coordinates.

$$\frac{\partial g_{ab}}{\partial x^c} = \frac{\partial \underline{e_a}.\underline{e_b}}{\partial x^c} = \frac{\partial \underline{e_a}}{\partial x^c}.\underline{e_b} + \underline{e_a}.\frac{\partial \underline{e_b}}{\partial x^c} = \Gamma^d{}_{ac}\underline{e_d}.\underline{e_b} + \Gamma^f{}_{bc}\underline{e_a}.\underline{e_f} = \Gamma^d{}_{ac}g_{db} + \Gamma^f{}_{bc}g_{af}$$

relabel to get:

$$\frac{\partial g_{ab}}{\partial x^c} = \Gamma^d{}_{ac}g_{db} + \Gamma^d{}_{bc}g_{ad}$$

then move all suffixes around to get

$$\frac{\partial g_{bc}}{\partial x^a} = \Gamma^d_{ba}g_{dc} + \Gamma^d_{ca}g_{bd}$$

$$\frac{\partial g_{ca}}{\partial x^b} = \Gamma^d_{cb}g_{da} + \Gamma^d_{ab}g_{cd}$$

add the first two and subtract the third to get an expression for the Christoffel symbols in terms of the derivities of the metric.

$$2\Gamma^d_{ca}g_{db} = \frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b}$$

so multiply by $\frac{1}{2}g^{fb}$ to get

$$\Gamma^f_{ca} = \frac{1}{2}g^{fb}(\frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b})$$

So we can calculate these horrors!!