(a) equivalence principle - gravity is the same as acceleration and/or that free fall is the same as no gravity [1 mark]

gravity=curvature. [2 marks]

spacetime curved by mass/energy [1 mark]

travel in straight line (natural paths) over curved surface so path looks curved. [1 mark]

(b) $\partial_c g_{ab} = \partial_c(\underline{e}_a.\underline{e}_b) = \partial_c(\underline{e}_a).\underline{e}_b + \underline{e}_a.\partial_c(\underline{e}_b)$ [1 mark]

$= \Gamma^d{}_{ca}\underline{e}_d.\underline{e}_b + \underline{e}_a\Gamma^d{}_{cb}\underline{e}_d = \Gamma^d{}_{ca}g_{db} + \Gamma^d{}_{cb}g_{ad}$ [1 mark]

$a \to b$, $b \to c$ and $c \to a$ gives $\partial_a g_{bc} = \Gamma^d{}_{ab}g_{dc} + \Gamma^d{}_{ac}g_{bd}$. Repeat $\partial_b g_{ca} = \Gamma^d{}_{bc}g_{da} + \Gamma^d{}_{ba}g_{cd}$ [1 mark]

$\partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca} = \Gamma^d{}_{ca}g_{db} + \Gamma^d{}_{cb}g_{ad} + \Gamma^d{}_{ab}g_{dc} + \Gamma^d{}_{ac}g_{bd} - \Gamma^d{}_{bc}g_{da} - \Gamma^d{}_{ba}g_{cd} = 2\Gamma^d{}_{ca}g_{db}$ [1 mark]

$g^{fb}g_{bd}\Gamma^d{}_{ca} = \delta^f{}_d\Gamma^d{}_{ca} = \Gamma^f{}_{ca} = \frac{1}{2}g^{fb}(\partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca})$ [1 mark]

(c)

$$\frac{D\chi}{du} = \frac{D(\mu_a\lambda^a)}{du} = \frac{D\mu_a}{du}\lambda^a + \mu^a\frac{D\lambda^a}{du}$$

ie Leibniz's rule for derivatives [1 mark]

but absolute derivative of scalar same as normal so $D\chi/du = d\chi/du = d(\mu_a\lambda^a)/du = (d\mu_a/du)\lambda^a + \mu_a d\lambda^a/du$ [1 mark]

$(d\mu_a/du)\lambda^a + \mu_a d\lambda^a/du = D\mu_a/du\lambda^a + \mu_a(d\lambda^a/du + \Gamma^a_{bc}\lambda^b dx^c/du)$

$(d\mu_a/du)\lambda^a = D\mu_a/du\lambda^a + \mu_a\Gamma^a_{bc}\lambda^b dx^c/du$ [1 mark]

relabel indices so cancel λ^b: $(d\mu_b/du)\lambda^b = D\mu_b/du\lambda^b + \mu_a\Gamma^a_{bc}\lambda^b dx^c/du$ [1 mark]

derivative - compare quantities across small spatial difference. but vector can change from spatial curvature as well as real change. so real change is total - space curvature (parallel transport). [1 mark]

(d) null metric for light $0 = (1-2m/r)c^2\dot{t}^2 - r^2\dot{\phi}^2$ [1 mark]

substitute in $\dot{t}^2 = r^3\dot{\phi}^2/(mc^2)$ so $0 = (1-2m/r)c^2r^3\dot{\phi}^2/(mc^2) - r^2\dot{\phi}^2$ [1 mark]

$0 = (1-2m/r)r^3 - mr^2 = r^3 - 2mr^2 - mr^2$ [1 mark]

$r^3 - 3mr^2 = 0$ so $r = 3m$ [1 mark]

unstable orbits [1 mark]

(e) particles $c^2 = (1-2m/r)c^2\dot{t}^2 - r^2\dot{\phi}^2$ [1 mark]

substitute in $\dot{t}^2 = r^3\dot{\phi}^2/(mc^2)$ so $c^2 = (1-2m/r)c^2 r^3\dot{\phi}^2/(mc^2) - r^2\dot{\phi}^2$

$mc^2 = (1-2m/r)r^3\dot{\phi}^2 - r^2 m\dot{\phi}^2$

$mc^2 = \dot{\phi}^2(r^3 - 3mr^2)$

$\dot{\phi}^2 = mc^2/(r^3(1-3m/r))$ [2 marks]

$\dot{\phi}^2 = (d\phi/d\tau)^2$ so for orbit at r_0 its $(2\pi/\tau)^2 = mc^2/(r_0^3(1-3m/r_0))$

so $\tau = 2\pi[r_0^3(1-3m/r_0)/(mc^2)]^{1/2}$ [2 marks]

(f) in the limit as $m \to 0$ then

$$\left(\frac{du}{d\phi}\right)^2 + u^2 \to \frac{E^2}{c^2 L_z^2}$$

[1 mark]

test solution $u = u_0 \sin\phi$ so $du/d\phi = u_0\cos\phi$ and $(du/d\phi)^2 = u_0^2\cos^2\phi$ [1 mark]

substitute in $u_0^2\cos^2\phi + u_0^2\sin^2\phi = E^2/(c^2L_z^2) = u_0^2$ [1 mark]

distance of closest approach is $r_0 = 1/u_0 = cL_z/E$ [1 mark]

paths get bent towards the massive object [1 mark]

U=unseen - but they will have seen similar examples using a different metric

(a) $L = \frac{1}{2}[r^{-2}\dot{r}^2 + r^2\dot{\phi}^2]$ [U: 1 mark]

$\partial L/\partial r = \frac{1}{2}[-2r^{-3}\dot{r}^2 + 2r\dot{\phi}^2] = -r^{-3}\dot{r}^2 + r\dot{\phi}^2$ [U: 1 mark]

$\partial L/\partial \dot{r} = \frac{1}{2}r^{-2}2\dot{r} = r^{-2}\dot{r}$ [U: 1 mark]

sub into EL: $\frac{d}{d\tau}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$

$d/d\tau(r^{-2}\dot{r}) + r^{-3}\dot{r}^2 - r\dot{\phi}^2 = 0$ [U: 1 mark]

$r^{-2}\ddot{r} + \dot{r}.(-2r^{-3})\dot{r} + r^{-3}\dot{r}^2 - r\dot{\phi}^2 = 0$

so $\ddot{r} - (1/r)\dot{r}^2 - r^3\dot{\phi}^2 = 0$ [U: 1 mark]

$\partial L/\partial \phi = 0$ [U: 1 mark]

$\partial L/\partial \dot{\phi} = r^2\dot{\phi}$ [U: 1 mark]

$d/d\tau(r^2\dot{\phi}) - 0 = 0$ [U: 1 mark]

$r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} = 0$ ie $\ddot{\phi} + (2/r)\dot{r}\dot{\phi} = 0$ [U: 1 mark]

(b) geodesic equation in r is

$$\ddot{r} + \Gamma^r{}_{rr}\dot{r}^2 + \Gamma^r{}_{r\phi}\dot{r}\dot{\phi} + \Gamma^r{}_{\phi r}\dot{\phi}\dot{r} + \Gamma^r{}_{\phi\phi}\dot{\phi}^2 = 0$$

$\ddot{r} + \Gamma^r{}_{rr}\dot{r}^2 + 2\Gamma^r{}_{r\phi}\dot{r}\dot{\phi} + \Gamma^r{}_{\phi\phi}\dot{\phi}^2 = 0$ [U: 2 marks]

and in ϕ is

$\ddot{\phi} + \Gamma^\phi{}_{rr}\dot{r}^2 + 2\Gamma^\phi{}_{r\phi}\dot{r}\dot{\phi} + \Gamma^\phi{}_{\phi\phi}\dot{\phi}^2 = 0$ [U: 2 marks]

so compare coefficients and the only non zero christoffel symbols in r are $\Gamma^r{}_{rr} = -1/r$ and $\Gamma^r{}_{\phi\phi} = -r^3$ [U: 1 mark]

and $\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = 1/r$ [U: 1 mark]

(c) $\phi = s/a - b$: $\dot{\phi} = 1/a$, $\ddot{\phi} = 0$, $\dot{r} = \ddot{r} = 0$ [U: 1 mark]

sub into EL for r: $-r^3(1/a)^2 \neq 0$ so NOT geodesic, [U: 1 mark]

even though EL in ϕ has $\ddot{\phi} + 2\Gamma^\phi{}_{r\phi}\dot{r}\dot{\phi} = 0 + (2/r).0.(1/a) = 0$

(d) $d\lambda^r/dr + \Gamma^r{}_{Br}\lambda^B = 0$ [U: 1 mark]

$d\lambda^r/dr + \Gamma^r{}_{rr}\lambda^r = d\lambda^r/dr - 1/r\lambda^r = 0$ [U: 1 mark]

$d\lambda^r/dr = \lambda^r/r$ so $d\lambda^r/\lambda^r = dr/r$ [U: 1 mark]

hence $[\log\lambda^r]_{\lambda^r(r_0)}^{\lambda^r(r_1)} = [\log r]_{r_0}^{r_1}$ [U: 1 mark]

and $\lambda^r(r_1)/\lambda(r0) = r_1/r_0$ [U: 1 mark]

similarly $d\lambda^\phi/dr + \Gamma^\phi{}_{Br}\lambda^B = 0$ [U: 1 mark]

$d\lambda^\phi/dr + \Gamma^\phi{}_{\phi r}\lambda^\phi = d\lambda^\phi/dr + 1/r\lambda^\phi = 0$ [U: 1 mark]

$d\lambda^\phi/dr = \lambda^\phi/r$ so $d\lambda^\phi/\lambda^\phi = -dr/r$ [U: 1 mark]

hence $[\log\lambda^\phi]_{\lambda^\phi(r_0)}^{\lambda^\phi(r_1)} = [-\log r]_{r_0}^{r_1}$ [U: 1 mark]

and $\lambda^\phi(r_1)/\lambda^\phi(r_0) = r_0/r_1$ [U: 1 mark]

length $g_{rr}\lambda^r(r_1)\lambda^r(r_1) + g_{\phi\phi}\lambda^\phi(r_1)\lambda^\phi(r_1)$ [U: 1 mark]

$= (r_1)^{-2}(\lambda^r(r_0))^2(r_1/r_0)^2 + (r_1)^2(\lambda^\phi(r_0))^2(r_0/r_1)^2$ [U: 1 mark]

$= \lambda^r(r_0)^2/(r_0)^2 + \lambda^\phi(r_0)^2(r_0)^2$ so no dependence on r_1 so its constant with respect to r_1. [U: 1 mark]

(a) $c^2 = (1-2m/r)c^2\dot{t}^2 - (1-2m/r)^{-1}\dot{r}^2$ and $(1-2m/r)\dot{t} = 1$ so

$c^2 = (1-2m/r)c^2/(1-2m/r)^2 - (1-2m/r)^{-1}\dot{r}^2$

$c^2(1-2m/r) = c^2 - \dot{r}^2$

$\dot{r} = dr/d\tau = \pm c\sqrt{2m/r}$ [S: 2 marks]

-ve as infall. $-dr\sqrt{r/2m} = cd\tau$

$$c\tau = \int_{r_1}^{r_2} -dr\sqrt{r/2m} = \int_{r_2}^{r_1} dr\sqrt{r/2m} = \frac{1}{\sqrt{2m}}\frac{[r_1^{3/2} - r_2^{3/2}]}{3/2}$$

[S: 2 marks]

This is continuous across $r = 2m$ [S: 1 mark]

(b) coordinate velocity dr/dt so back to metric and divide by dt^2 to get $c^2/\dot{t}^2 = (1-2m/r)c^2 - (1-2m/r)^{-1}(dr/dt)^2$ [S 1 mark]

and then substitute in $(1-2m/r)\dot{t} = 1$ to get $c^2(1-2m/r)^2 = (1-2m/r)c^2 - (1-2m/r)^{-1}(dr/dt)^2$ [S: 2marks]

$c^2(1-2m/r)^3 = (1-2m/r)^2c^2 - (dr/dt)^2$

$(dr/dt)^2 = c^2(1-2m/r)^2[1-(1-2m/r)] = c^2(1-2m/r)^2 2m/r$

hence $(dr/dt) = \pm c(1-2m/r)\sqrt{2m/r}$ [S: 2 marks]

inwards. $(dr/dt) = -c(1-2m/r)\sqrt{2m/r}$, and $r = 2m+\epsilon$ so $(dr/dt) = -c(1-2m/(2m+\epsilon)\sqrt{2m/(2m+\epsilon)}$

$(dr/dt) = -c(1-(1+\epsilon/2m)^{-1})(1+\epsilon/2m)^{-1/2}$ [U 2 marks]

binomial $(dr/dt) \to -c(1-(1-\epsilon/2m))(1-\epsilon/4m) \to -c\epsilon/2m$[U: 2 marks]

$$ct = -\int_{6m}^{2m+\epsilon} \frac{dr}{1-2m/r}\sqrt{\frac{r}{2m}} \to -\int_{6m}^{2m+\epsilon} \frac{2m}{\epsilon}d\epsilon \to -2m\log\epsilon \to \infty \text{ as } \epsilon \to 0$$

[S: 2 marks]

(c) new metric is $c^2d\tau^2 = (1-2m/r)(cdT - \frac{(2m/r)^{1/2}}{(1-2m/r)}dr)^2 - (1-2m/r)^{-1}dr^2$

$$c^2d\tau^2 = (1-2m/r)c^2[dT^2 - 2\frac{(2m/r)^{1/2}}{(1-2m/r)}drdT + \frac{(2m/r)}{(1-2m/r)^2}dr^2] - (1-2m/r)^{-1}dr^2$$

[U 2 marks]

$$c^2d\tau^2 = (1-2m/r)c^2dT^2 - 2(2m/r)^{1/2}cdTdr + (2m/r)(1-2m/r)^{-1}dr^2 - (1-2m/r)^{-1}dr^2$$

[U 2 marks]

$$c^2d\tau^2 = (1-2m/r)c^2dT^2 - 2(2m/r)^{1/2}cdTdr - dr^2$$

[U 2 marks]

light has $d\tau = 0$ so $0 = (1-2m/r)c^2dT^2 - 2(2m/r)^{1/2}cdTdr - dr^2$

hence $0 = (1-2m/r)c^2 - 2(2m/r)^{1/2}cdr/dT - (dr/dT)^2$ [U 2 marks]

solve quadratic

$dr/dT = [-2(2m/r)^{1/2}c \pm \sqrt{4.2mc^2/r - 4(1-2m/r)c^2}]/2$ [U 2 marks]

$= -2m/r^{1/2} \pm \sqrt{2mc^2/r - (1-2m/r)c^2} = -2mc/r^{1/2} \pm c$

$= (-\sqrt{2m/r} \pm 1)c$ [U 2 marks]

outgoing have $dr/dT = (1 - \sqrt{2m/r})c < 0$ for $r < 2m$ so the light goes inwards even though it was emitted outwards! [U 2 marks]