6.7 Geodesic paths - photon orbits (cont)

one of the Newtonian predictions is that photons should travel in a straight line. The equation for this in terms of u and ϕ would be $u = u_0 \sin\phi$ where u_0 is the value of u at closest approach, $r_0 = 1/u_0$ which is also called an impact parameter. Then $du/d\phi = u_0 \cos\phi$ so $(du/d\phi)^2 = u_0^2 \cos^2\phi$ and $(du/d\phi)^2 + u^2 = u_0^2 \cos^2\phi + u_0^2 \sin^2\phi = u_0^2$. i.e. its the same as the GR except GR has an extra term $2mu^3$. This is very small, so here we make this explicit by saying $2m = \epsilon$. then

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{E^2}{c^2 L_z^2} + \epsilon u^3$$

at closest approach then $u = u_0$ again and $du/d\phi = 0$ so

$$u_0^2 = \frac{E^2}{c^2 L_z^2} + \epsilon u_0^3$$

we can use this to substitute for $E^2/c^2 L_z^2 = u_0^2 - \epsilon u_0^3$ in the general equation

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = u_0^2(1 - \epsilon u_0) + \epsilon u^3$$

This should have a solution which is just a small perturbation of the flat spacetime solution - maybe $u = u_0 \sin\phi + \epsilon f(\phi)$ where f is some function of ϕ to be determined. Stick this in, and ignore anything with higher powers of ϵ

$$2\cos\phi \frac{df}{d\phi} + 2f \sin\phi = u_0^2(\sin^3\phi - 1)$$

This is just a first order differential equation of f, so we'll go look up the solution.

$$f = \frac{1}{2} u_0^2 (1 + \cos^2\phi - \sin\phi) + A\cos\phi$$

A is a constant of integration so can be found from the boundary conditions. At $r = \infty$ then $\phi = 0$ and $f = 0$, so $A = -u_0^2$. So our full solution for the weak field effect of gravity on the path of a light beam is

$$u = u_0(1 - \frac{1}{2}\epsilon u_0)\sin\phi + \frac{1}{2}\epsilon u_0^2 (1 - \cos\phi)^2$$

So our full solution for the weak field effect of gravity on the path of a light beam is

$$u = u_0(1 - \frac{1}{2}\epsilon u_0)\sin\phi + \frac{1}{2}\epsilon u_0^2(1-\cos\phi)^2$$

so we no longer expect a straight line as the photon goes from $\phi = 0$ to π. As $r \to \infty$ i.e. $u \to 0$ then $\phi \to \pi + \alpha$ where $\alpha \ll 1$. $\cos(\pi+\alpha) = \cos\pi\cos\alpha - \sin\pi\sin\alpha \approx -1$ while $\sin(\pi+\alpha) = \sin\pi\cos\alpha + \cos\pi\sin\alpha = -\sin\alpha \approx -\alpha$.

$$0 = -(1 - \frac{1}{2}\epsilon u_0)\alpha + 2\epsilon u_0 \approx -\alpha + 2\epsilon u_0$$

$$\alpha \sim 2\epsilon u_0 = 4\frac{GM}{r_0 c^2}$$

Take r_0 as equal to the suns radius and look at a star in the line of sight just away from the limb of the sun and then the prediction for the deflection of its position is 1.75 seconds of arc. This was the first test of GR, measuring positions of stars close to the sun as seen during a solar eclipse and comparing these to their positions as seen when the sun was nowhere near the line of sight.

6.8 Orbits in strong field GR

We did wimpy weak field geodesics for particles and photons. Now lets do it in full strong field GR

$$ds^2 = c^2 d\tau^2 = (1-2m/r)c^2 dt^2 - (1-2m/r)^{-1}dr^2 - r^2 d\phi^2$$

cyclic coordinates $(1-2m/r)\dot{t} = E/c^2$ and $r^2\dot{\phi} = L_z$ substitute back into it the metric and solve for $\dot{r}$

$$c^2 = (1-2m/r)c^2\dot{t}^2 - (1-2m/r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$

$$c^2(1-2m/r) = E^2/c^2 - \dot{r}^2 - L_z^2(1-2m/r)/r^2$$

$$\dot{r}^2 = E^2/c^2 - c^2(1 + L_z^2/c^2r^2)(1-2m/r)$$

$$\dot{r}^2 = c^2[E^2/c^4 - V^2(r)]$$

where $V^2(r) = (1 + L_z^2/c^2r^2)(1-2m/r)$. this is the effective potential - the terms represent an angular momentum barrier and gravity.

We can get more insight into this by contrasting with Newtonian - angular momentum and gravity give an effective potential. gravity is easy. angular momentum needs a bit of thinking. centrifugal force is $F = -mv^2/r$. but $L_z = r^2\dot{\phi} = rv$ so $F/m = -L_z^2/r^3$. Potential is the integral of force so this gives a potential term of $= L_z^2/2r^2$. hence in newtonian we get

$$V(r) = \frac{L_z^2}{2r^2} - GM/r$$

The first term is the angular momentum barrier - particle orbits around a large mass - it wants to get closer because gravity is attractive, but as it gets closer then by conservation of angular momentum it goes faster so there is a bigger force outwards. This $V(r) \to \infty$ as $r \to 0$, and $\to 0$ as $r \to \infty$ and has a minimum. If the particle energy $E = V_{min}$ then the orbit is circular. if $V_{min} < E < 0$ then there is a range of r from r_{min} to r_{max} which the particle can access - this is an elliptical orbit. And if instead $E > 0$ then it is not bounded at large r, but only limited at small r by the angular momentum barrier - this is a hyperbolic orbit.

In GR then the potential is a bit diffferent to the Newtonian shape. V^2 - zero at $r = 2m$, $\to -\infty$ as $r \to 0$ and $\to 1$ as $r \to \infty$. turning points where $dV^2/dr = 0$.

$$(1 + L_z^2/c^2r^2) - 2md(r^{-1})/dr + (1 - 2m/r)L_z^2/c^2 d(r^{-2})/dr = 0$$

$$(1 + L_z^2/c^2r^2)2m/r^2 = 2L_z^2/c^2r^3(1 - 2m/r)$$

$$(1 + L_z^2/c^2r^2)mr^2 = L_z^2 r/c^2(1 - 2m/r)$$

$$mr^2 + mL_z^2/c^2 = L_z^2 r/c^2 - 2mL_z^2/c^2$$

$$r^2 - L_z^2 r/mc^2 + 3L_z^2/c^2 = 0$$

$$r = [L_z^2/mc^2 \pm \sqrt{L_z^4/m^2c^4 - 4.3L_z^2/c^2}]/2$$

$$r = L_z^2/2mc^2 \pm L_z^2/2mc^2\sqrt{1 - 12m^2c^2/L_z^2}$$

So in general has 2 turning points. Do d^2V^2/dr^2 and see that the maximum is at smaller r, minimum at larger r.

But for $L_z = \sqrt{12}mc$ then the two turning points merge together, with both at $r = 6m$. $d^2V^2/dr^2 = 0$ so this is neither a maximum nor a minimum - its a point of inflection. Stable orbits require a MINIMUM in the potential and that the particle energy E^2 is less than V^2 on either side of the minimum so that the particle is confined to a range of radii (elliptical orbit). If instead the particle energy is exactly at the minimum of the potential then there is no range in radii accessible to the particle and its a circular orbit. If E^2 is such that there is a potential barrier at small r, but NOT at larger r then this is an unbound hyperbolic orbit. If $E^2 > V^2_{max}$ then the particle can get everywhere, even to $r = 0$ and hence it hits the black hole (and never comes out!!)

Anyway, for a point of inflection there is no minimum. so this is NOT a stable orbit. Its unstable - any small perturbation and the particle will spiral in towards the black hole. $r = 6m$ is the last stable orbit around a Schwarzschild black hole. It is NOT where a particle would orbit around at the speed of light - thats at $r = 3m$. There are orbits that are possible between $r - 6m$ and 3m but THEY ARE NOT STABLE. Gravity is just stronger in GR than in Newtonian - the centrifugal barrier is simply not big enough to hold up any more.

7 Time dilation in circular orbits

Then our metric simplifies a bit as $dr = 0$

$$ds^2 = c^2 d\tau^2 = c^2(1 - 2m/r)dt^2 - r^2 d\phi^2$$

so the Lagrangian is $L = 1/2(c^2(1 - 2m/r)\dot{t}^2 - r^2\dot{\phi}^2)$ We can do the Euler lagrange equations for r to get (see homework) $mc^2\dot{t}^2 = r^3\dot{\phi}^2$

so the COORDINATE time for one orbit is

$$mc^2 t_{orb}^2 = r^3(2\pi)^2 \quad t_{orb} = 2\pi(r^3/GM)^{1/2}$$

ie same as Newtonian except this is COORDINATE time, not proper time.

PROPER TIME measured by someone going round on this orbit (sub. back into metric)

$$c^2 = c^2(1 - 2m/r)\dot{t}^2 - r^2\dot{\phi}^2 = c^2(1 - 2m/r)\dot{t}^2 - r^2 mc^2\dot{t}^2/r^3$$

$$1 = (1 - 3m/r)(dt/d\tau)^2$$

$$d\tau = (1 - 3m/r)^{1/2} dt$$

$$\tau_{orb} = (1 - 3m/r)^{1/2} t_{orb}$$

So proper time goes to zero at $r = 3m$, and goes COMPLEX for $r < 3m$. What does this MEAN physically ? it means that there are NO geodesic circular orbits possible at $r < 3m$. This is very different to Newtonian gravity where circular orbits are always possible. In Newtonian orbits you can always balance gravity just by running round faster. In SR/GR you can only run as fast as the speed of light. $r = 3m$ is the orbit for going at the speed of light. So there are no faster orbits. And if you were going around on this orbit, then you wouldn't age.

But this is different also to the proper time as experienced by someone hovering at this radius r - held there by firing rockets (not on a geodesic). We know how to relate proper time to coordinate time when we are stationary - its simply the metric with no spatial parts as all the $dr, d\theta, d\phi = 0$ i.e. $d\tau = (1 - 2m/r)^{1/2} dt$.

Suppose there are 2 spacecraft at $r = 6m$, one HOVERING, the other orbiting. The proper time as measured for one orbit for the hovering craft is $\tau_{hov} = (1-2m/r)^{1/2} t_{orb} = (2/3)^{1/2} t_{orb}$ while the proper time measured ON the orbiting spacecraft is $\tau_{orb} = (1 - 3m/r)^{1/2} t_{orb} = (1/2)^{1/2} t_{orb}$ - the coordinate time intervals are the same (see the gravitational redshift section). Then the ratio of proper times is simply $\tau_{orb}/\tau_{hov} = (1/2)^{1/2}/(3/2)^{1/2} = \sqrt{3}/2 = 0.866$. The astronauts in the hovering probe see a LONGER elapsed time - they are older than their companions who orbited.