4.2   Interpretation of Ψ

What can we do with a wave function? what does it tell us? The fact that we needed to use complex numbers in our justification for the Schroedinger equation shows immediately that we should not attempt to give wavefunctions a physical existence in the same way as water waves have a physical existance. complex quantities cannot be measuered by any actual physical instrument. so we don't have to ask the questions what is waving and what is it waving in. And its questions like these that led to the aether in electromagnetism! but because our wave function here can be complex we are not tempted to make the same mistake again. wavefunctions are computational devices which have significance only in the context of the Schroedinger theory of which they are a part (this is a cop out, which we will look at in more detail in the final lecture!)

The wave function contains all the information which the uncertainty principle allows us to know. What is this information?

Going back to the double slit experiment, but turn down the intensity of the electon source so that only one electron comes through at a time. we can see the single hits on the screen where each electron lands (showing that individual electrons really are going through the slits). But after a while, when many 'single' particles have built up a pattern, we get the characteristic pattern of interference. Interference does not occur between electrons, but is a property of a single electron.

One single measurement does not have a predictable result - the electron hit can be anywhere in the pattern. only when we consider many identical systems do we get the full pattern. This suggests that for an individual particle the process is of a statistical nature, so it is telling us something about the probability that the particle will hit the screen at a certain point.

Since our wavefunction is complex, and probabilities must be real, this suggests that we associate a probability $\propto |\Psi(x,t)|^2$. This would be similar to classical waves where the intensity of a wave $I \propto \text{Amplitude}^2$. Then the probability of finding a particle between x and x+dx is

$$P(x,t)dx \propto |\Psi(x,t)|^2 dx = \Psi^*(x,t)\Psi(x,t)dx$$

where the position probability distribution function or probability density $|P(x,t)|^2$ has units 1/length and the wavefunction $\Psi(x,t)$ has units $1/(\text{length})^{1/2}$. This is always real even when our wavefunction is imaginary. Suppose $\Psi = a + ib$ where a, b are real. Then $\Psi^*\Psi = (a - ib)(a + ib) = a^2 + b^2$.

Thus the interpretation of the wavefunction is a statistical one. we talk about the wavefunction of an individual particle but it is more useful to think about it as describing the behaviour expected from an ensemble of identical systems.

If we are interpreting $|\Psi(x,t)|^2$ as relating to probability of detection of the particle, then this means we have a normalisation condition since there is unity probability that the particle is SOMEWHERE i.e. if we integrate over all space $\int_{-\infty}^{+\infty}|\Psi(x,t)|^2dx = 1$

So, lets go into this with our 1D Schrodinger equation for a free particle $\Psi(x,t) = Ae^{i(kx-\omega t)}$. Then

$$\int_{-\infty}^{+\infty}\Psi^*(x,t)\Psi(x,t)dx = A^2\int_{-\infty}^{+\infty}e^{-i(kx-\omega t)}e^{i(kx-\omega t)}dx = A^2\int_{-\infty}^{+\infty}dx = \infty$$

Thats not a good start. Its because a plane wave along the x axis is completely delocalised - there is constant probability to find it anywhere along the x-axis. Our plane wave has definite momentum $p = \hbar k$ in the x-direction so $\Delta p = 0$ hence Δx is ∞ because of the Heisenburg uncertainty principle.

In practice we'd confine it by the experiment to a box of length $L >> \lambda$ (e.g. the size of the room!) so

$$A^2\int_0^L e^{-i(kx-\omega t)}e^{i(kx-\omega t)}dx = A^2L = 1$$

so then $A^*A = 1/L$ so we are free to choose this to be any (complex) number where $A^2 = 1/L$ eg $A = 1/\sqrt{L}$ or $A = -1/\sqrt{L}$ or $A = i/\sqrt{L}$ or $A = -i/\sqrt{L}$ or.... so pick the one which makes life easy and go for $A = 1/\sqrt{L}$. we know that we don't have a physical meaning for the wavefunction itself, only its square, so this phase doesn't matter.

so our normalised wavefunction is $\Psi(x,t) = 1/\sqrt{L}e^{i(kx-\omega t)}$. The probability of finding it in any section from x to $x + dx$ is $\Psi^*(x,t)\Psi(x,t)dx$ and $\Psi^*(x,t)\Psi(x,t)dx = dx/L$ so there is equal probability of finding a particle with this wavefunction anywhere in the box of size L

4.3   time independent Schroedinger equation

If $U(x,t) = U(x)$ i.e. the potential is independent of time then this simplifies as the wavefunction is separable $\Psi(x,t) = \psi(x)T(t)$ e.g the free particle solution is $\Psi(x,t) = Ae^{i(kx-\omega t)} = Ae^{ikx}e^{-i\omega t} = \psi(x)e^{-iEt/\hbar}$

Put $\Psi(x,t) = \psi(x)T(t)$ into the equation and:

$$\frac{-\hbar^2}{2m}\frac{\partial^2[\psi(x)T(t)]}{\partial x^2} + U(x)\psi(x)T(t) = i\hbar\frac{\partial[\psi(x)T(t)]}{\partial t} = E\psi(x)T(t)$$

$$\frac{-\hbar^2}{2m}T(t)\frac{d^2\psi}{dx^2} + U(x)\psi(x)T(t) = i\hbar\psi(x)\frac{dT}{dt}E\psi(x)T(t)$$

Divide by $\psi(x)T(t)$

$$\frac{-\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} + U(x) = i\hbar\frac{1}{T(t)}\frac{dT}{dt} = E$$

The LHS is a function only of x, while the RHS is a function only of t. the only way these can be equal to each other is if NEITHER is a function of x or t i.e. if this is equal to a constant. This separation constant is E so

$$E = i\hbar\frac{1}{T(t)}\frac{dT(t)}{dt}$$

$$\frac{E}{i\hbar}dt = -iE/\hbar dt = dT(t)/T(t)$$

cf the standard derivative $d(e^{ct})/dt = ce^{ct}$ so $T(t) = e^{-iEt/\hbar}$ on the other side we have

$$E = \frac{-\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} + U(x)$$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

The wave function $\Psi(x,t)$ for a state of definite energy E is the product of a time independent wavefunction $\psi(x)$ and its time dependence which is $e^{-iEt/\hbar}$.

States of definite energy are called stationary states because their probability distribution function is NOT dependent on time

$$|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t) = \psi^*(x)[e^{-iEt/\hbar}]^*\psi(x)e^{-iEt/\hbar}$$

$$= \psi^*(x)e^{iEt/\hbar}\psi(x)e^{-iEt/\hbar} = \psi^*(x)\psi(x) = |\psi(x)|^2$$