1.12 relativisitic energy

So now lets try to get to an understanding of energy in a relativisitc context. We know that energy is force times distance, so when we are accelerating from rest with the force in the same direction as the direction of motion then the work done in going from 0 to some distance x is

$$W = \int_0^x F dx = \int_0^x m\gamma^3 a dx$$

we can break this up as $a_x dx = dv_x/dt dx = dv_x dx/dt = v_x dv_x$

$$W = \int_0^x m\gamma^3 a dx = \int_0^v m\gamma^3 v_x dv_x = m \int_0^v \frac{v_x}{(1 - v_x^2/c^2)^{3/2}} dv_x$$

assuming that we start at from rest at $x = 0$.

lets do a subsitution with $\alpha = 1 - v_x^2/c^2$ to make this integral nicer. then differentiate to get $d\alpha = -2v_x dv_x/c^2$ so $v_x dv_x = -\frac{1}{2}c^2 d\alpha$. We have to remember to change the integral limits as well, so the lower limit is $v_x = 0$ which gives $\alpha = 1$, while the upper limit is whatever α corresponds to the end velocity we accelerate too. Hence

$$W = -\frac{m}{2} \int_1^\alpha c^2 \alpha^{-3/2} d\alpha$$

$$= \frac{-mc^2}{2} \Big[\frac{\alpha^{-1/2}}{-1/2}\Big]_1^\alpha$$

$$= mc^2(\alpha^{-1/2} - 1) = mc^2\Big(\frac{1}{(1 - v^2/c^2)^{1/2}} - 1\Big) = mc^2(\gamma - 1)$$

so relativisitic kinetic energy required to accelerate something from rest to velocity v is $K = mc^2(\gamma - 1)$. Lets look at this in the limits - when the particle is a rest we get KE=0 as expected. then for $v \ll c$ we get an expansion

$(\gamma - 1) = (1 - \beta^2)^{-1/2} - 1 = 1 + (-1/2)(-\beta^2) \ldots - 1 = v^2/(2c^2)$

so then $K \approx mc^2 v^2/(2c^2) = 1/2mv^2$ which is the classical value. But as $v \to c$ then the relativistic and classical KE diverge. and the relativisitc one is correct!

so kinetic energy is properly given as $(\gamma - 1)mc^2$ - but this is the difference between two terms, $\gamma mc^2 - mc^2$.

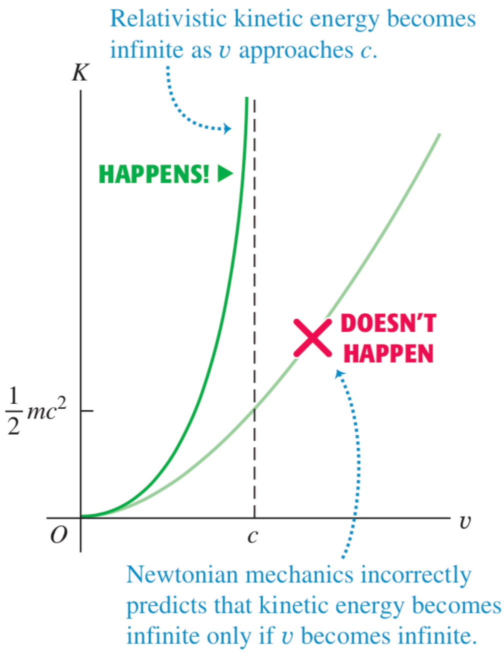

Figure 9:

the second term exists even when the particle is at rest. this is the rest energy of the particle. so we can also define total energy $E = K + mc^2 = \gamma mc^2$

now we can see that we can accelerate and accelerate and the KE goes up and up. but the speed cannot go faster than the speed of light.

we can relate the particle energy directly to the momentum

$E = \gamma mc^2$ and $p = \gamma mv$

re-write these as $E/mc^2 = \gamma = (1-\beta^2)^{-1/2}$ so $(E/mc^2)^2 = 1/(1-\beta^2)$ while $p/mc = \gamma\beta$ so $(p/mc)^2 = \gamma^2\beta^2$

$$\left(\frac{E}{mc^2}\right)^2 - \left(\frac{p}{mc}\right)^2 = \frac{1}{1-\beta^2} - \frac{\beta^2}{1-\beta^2} = \frac{1-\beta^2}{1-\beta^2} = 1$$

$E^2 - p^2m^2c^4/(m^2c^2) = m^2c^4$

$E^2 - p^2c^2 = m^2c^4$ or $E^2 = (pc)^2 + (mc^2)^2$

this is amazing as a particle with no rest mass can still have energy and momentum!

1.13 relativisic collisions and kinematics

Example YF37.77 - kaon production! some of the incident KE is used to create rest mass energy of the new particles so this is NOT an elastic collision!

$$p + p \to p + p + K^+ + K^-$$

The rest energy of each Kaon is 493.7MeV, the proton is 938.3MeV.

Calculate the minimum kinetic energy for proton 1 which allows this to occur if proton 2 is initially at rest.

This is much easier to do in the centre of mass frame. then the 2 protons have equal and opposite velocities and EVERYTHING is at rest afterwards

conserve energy: before $\gamma(u)m_pc^2 + \gamma(-u)m_pc^2 = 2\gamma(u)m_pc^2$

after $2m_pc^2 + 2m_Kc^2$. so then we can solve for $\gamma(u)$ as

$\gamma(u) = 1 + m_K/m_p = 1 + 493.7/938.3 = 1.53$ so $1 - \beta^2 = 1/1.53^2$ and $\beta = 0.76$.

no we nede to transform this to a velocity on one of the protons, so its at rest with respect to the other one.

pick the one moving to the right, make this S' so we have our standard setup for $u = +0.76c$. in our central frame, we had $v_x = 0.76c$ for the particle on the left, and $v_x = -0.76c$ for the particle on the right.

transform to the primed frame, so we see what the stationary particle sees

particle on the left:

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2} = 0$$

this is as expected as we wanted to be in the rest frame of the particles on the left!

particle on the right - this is what we want!

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2} = -0.76c{-}0.76c/(1{-}(-0.76c)(0.76c)/c^2 = -1.51/1{+}0.57 = -0.96c$$

so the particle on the right has very high velocity in the rest frame of the particle on the left. so kinetic energy $(\gamma - 1)m_pc^2 = (3.68 - 1)m_pc^2 = 2.68.3938MeVc^2 = 2515MeV$

Another inelestic collision

A particle of rest mass 1MeV/c^2 and kinetic energy 2MeV collides with a stationary particle of rest mass 2MeV/c^2. after collision the particles stick together

(a) total energy of particle 1 E=KE+rest massc^2 =3MeV

(b) total energy of particle 2 is just rest mass so $E = mc^2 = 2MeV/c^2 \times c^2 = 2MeV$

(c) total energy of system before collision is 5MeV

(d) what is the speed of the first particle before the collision

$KE = (\gamma - 1)mc^2$ so $2MeV = (\gamma - 1)1MeV/c^2 \times c^2$

$2 = \gamma - 1$ so $\gamma = 3$ and $\beta^2 = 1 - 1/\gamma^2$ so $\beta = \sqrt{8/9}$

(e) what is γ

$\gamma = 1/\sqrt{1-\beta^2} = 3$ - could also get this from total energy $E = \gamma mc^2$ and we know $E = 3MeV/c^2$ and $mc^2 = 1MeV/c^2$ so then $\gamma = 3$

(f) initial total momentum of the system - $p = \gamma mv = \gamma mc\beta = 3\sqrt{8/9}mc = \sqrt{8}(1MeV/c^2)c = \sqrt{8}MeV/c$

or we could do it from $E^2 = p^2c^2 + m^2c^4$ so $p^2 = E^2/c^2 - m^2c^2$

$p^2 = 9 - 1 = 8$ so $p = \sqrt{8}MeV/c$

(g) rest mass of system after collison

CONSERVE ENERGY!! total energy is $E_{final} = E_{initial} = 5MeV$

CONSERVE MOMENTUM $p_{before} = \gamma m_1 v = \sqrt{8}MeV/c = p_{after}$

so we know energy and momentum so we get rest mass from

$$E^2 = p^2c^2 + m^2c^4$$

so $5^2MeV^2 = 8MeV^2 + m^2c^4$ and $mc^2 = \sqrt{25-8} = \sqrt{17} = 4.1MeV$

so mass is $m = 4.1MeV/c^2$.

(h) speed of nucleus after collision

$p_{final} = \gamma mv$ so $\sqrt{8}MeV/c = \gamma mcv/c = \gamma\beta mc^2/c$

$$\sqrt{8}MeV/c = \gamma\beta(4.1MeV/c^2c^2)/c = \gamma\beta 4.1MeV/c$$

$\gamma\beta = \sqrt{8}/4.1 = 0.689$

$\beta^2 = 0.689^2(1-\beta^2)$ and rearrange to solve for $\beta^2 = 0.689^2/(1+0.689^2)$ which gives $\beta = 0.32$.

so $\gamma = 1.056$. check that we conserve energy! $5 = \gamma mc^2 = 1.0564.1MeV/c^2c^2 = 4.33MeV$ which does NOT equal 2+1=3MeV which is the initial rest mass energy of the two particles. some energy has been transformed into mass of the single 'stuck together' nucleus in this collision