1.13.1 Compton scattering

We saw that $E^2 = (pc)^2 + (mc^2)^2$ implies that particles which do not have mass can still have energy and momentum!! clearly this means that $p = \gamma m v$ is not the only way to define momentum, and in fact we'll see how to do this for massless particles (light!) in the next sections of the course. But in the meantime we are going to use it!

Compton scattering. photon, p, E, hits electron (rest mass m_e). electron recoils with p_e, E_e, photon bounces back p', E'.

Conservation of momentum (electron at rest so 0) $p+0 = p_e - p'$ - includes directions as well as magnitude as just in 1D. so $p_e = p + p'$

conservation of energy $E + m_e c^2 = E' + E_e$ so $E_e = E - E' + m_e c^2$

but we know that $E_e^2 = m^2 c^4 + p_e^2 c^2$ so

$$E_e^2 = m^2 c^4 + p_e^2 c^2 = [(E - E') + m_e c^2]^2$$

but we know that for photons $E = pc$ and $E' = p'c$ so we have

$$m^2 c^4 + p_e^2 c^2 = [(p - p')c + m_e c^2]^2 = (p - p')^2 c^2 + 2(p - p') m_e c^3 + m_e^2 c^4$$

$$p_e^2 = (p - p')^2 + 2(p - p') m_e c$$

go back to the electron momentum and square it $p_e^2 = (p + p')^2 = p^2 + (p')^2 + 2pp'$

then substitute in

$$p^2 + (p')^2 + 2pp' = (p - p')^2 + 2(p - p') m_e c = p^2 + (p')^2 - 2pp' + 2(p - p') m_e c$$

$$4pp' = 2pm_e c - 2p' m_e c$$

$$p'(2p + m_e c) = pm_e c$$

$$p' = \frac{pm_e c}{2p + m_e c}$$

We'll do this in particle physics units to show how this works!

the initial photon has energy E=1.25MeV, this corresponds to $p = E/c$ so momentum is $p = 1.25 MeV/c$.

the electron mass is $0.51 MeV/c^2$

so the final photon momentum is $p' = 1.25/c \times 0.51c/c^2/(2 \times 1.25/c + 0.51c/c^2) = 0.21 MeV/c$

and energy $E' = pc = 0.21$ MeV

1.14 Spacetime diagrams

I promised you that we would look at what it all means! lets draw spacetime diagrams, with ct as the y axis and x as the x axis. in my frame I am stationary, so I just move in time.

someone moving relative to me at a constant speed makes a straight line on the plot, where higher speed gives larger angle to the y axis.

light rays going to the right make a 45 degree angle to both x and ct, light rays going to the left make a 45 degree angle to -x and ct.

click on the transformation video - see 4:29 for newtonian motion

and 6:13 for light speed.

now lets spin this around, and rather than keeping our axes straight, lets plot them. we know

$t' = \gamma(t - ux/c^2)$ or $ct' = \gamma(ct - \beta x)$

and $x' = \gamma(x - ut)$ or $x' = \gamma(x - \beta ct)$

so if we want to draw this it makes more sense to sqeeze the axes rather than the space between them! lets do an example to make this specific. suppose $u = 0.5c$ so $\beta = 0.5$ and $\gamma = 1.15$

the x' axis starts at $ct = ct' = 0$ and has $ct' = 0$ so its given by $0 = \gamma(ct - \beta x)$ i.e. $ct = \beta x$

the t' axis starts at $x = x' = 0$ and then has $x' = 0$ so its given by $0 = \gamma(x - \beta ct$ so its equation is $x = \beta ct$.

so now we can draw this and see that the angle between x and x' and the angle between t and t' is given by $\tan\theta = \beta$.

so as we get closer and closer to the speed of light, we squeeze our axes closer and closer to the light cone.

This gives a nice way to visualise simulteneity - just dot on a line of constant ct and another line of constant ct' to see that they are different!!

1.15 Invariant interval

Time and space intervals are not absolute - they depend on the frame of the person measuring them! but there is a sneaky combination which IS absolute, so it must be more fundamental.

take the Lorentz transformations, and go to the limit of small time and space intervals

$$dx' = \gamma(dx - udt) \quad dy' = dy \quad dz' = dz \quad dt' = \gamma(dt - dxu/c^2)$$

Have a look at $c^2(dt')^2 - (dx')^2 - (dy')^2 - (dz')^2$
$= c^2\gamma^2(dt - dxu/c^2)^2 - \gamma^2(dx - udt)^2 - dy^2 - dz^2$
$= c^2\gamma^2(dt^2 - 2u/c^2 dxdt + u^2/c^4 dx^2) - \gamma^2(dx^2 - 2udxdt + u^2dt^2) - dy^2 - dz^2$
$= c^2dt^2 - dx^2 - dy^2 - dz^2$

so there is some interval which is a combination of time and space which is the same in any frame. what is this interval if we were in the rest frame?

rest frame is S'. if measuring time between two events at the same place then $dx' = dy' = dz' = 0$ and this interval is $c^2(dt')^2$ where $dt' = T_0$ is proper time!

so we could say $c^2dT^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

in the rest frame, we also have proper length so if we measure the ends simultaneously we'd get $(dx')^2 + (dy')^2 + (dz')^2$

so $ds^2 = c^2(dt')^2 - (dx')^2 - (dy')^2 - (dz')^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

where ds is SPACE-TIME interval. It is the same in any frame!

so pull it all together and we have $c^2dT^2 = ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2(dt')^2 - (dx')^2 - (dy')^2 - (dz')^2$ as an invarient

if we are in the rest frame then $dT = dt'$ and we have

$$c^2 = c^2 - (v'_x)^2 - (v'_y)^2 - (v'_z)^2$$

so we have a fixed speed - the speed of light - through SPACETIME.

the more of that speed we put through SPACE, the less there is left to travel through TIME, so we travel through time more slowly - we age less.

if you are interested in understanding more try 'The elegant Universe' by Brian Greene

and actually, I learnt some useful stuff from the childrens book series by Russell Stannard, 'The space and time of Uncle Albert' and 'Uncle Albert and the black holes'

one of the amazing insights I got from this was why you can't go past the speed of light! we have just done $E = mc^2$ which says mass and energy are interchangeable. so if we have a particle and give it some kinetic energy then in some sense we have added to its inertial mass... and when the kinetic energy starts to dominate over the rest mass then we are in trouble. we accelerate it, which increases its KE, which makes its mass increase, so its harder to accelerate... (it gets a bit nasty, as velocity has a direction whereas mass doesn;t so they don't have quite the same properties. but its a helpful way to see that whatever the factor is which scales between mass and energy, then this will be the speed limit.