

1 Special Relativity

Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}} \quad \text{where } \beta = u/c$$

Standard frame setup: PRIME IS THE REST FRAME OF THE PARTICLE WHICH MOVES WITH VELOCITY u

Time dilation $t = \gamma t'$ and length contraction $L = L'/\gamma$

It will do you no harm to learn also the Lorentz transformations

$$x' = \gamma(x - ut) \quad y' = y \quad z' = z \quad t' = \gamma(t - xu/c^2)$$

and velocity transformations

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2}$$

Relativistic mechanics: momentum $p = \gamma mv$, total energy $E^2 = (pc)^2 + (mc^2)^2$. so for particles $E = \gamma mc^2$ and kinetic energy $K = (\gamma - 1)mc^2$. For photons $m = 0$ so $E = pc = hf$

2 towards Quantum mechanics: photons

Photons behave as particles as well as waves: Einstein - photoelectric.

$$E = hf = hc/\lambda \quad \text{momentum } p = hf/c = h/\lambda$$

$$\text{photoelectric effect } K = eV = hf - \phi$$

ϕ is the work function (characteristic of each element) and electron kinetic energy K is measured from the stopping potential V .

3 towards quantum mechanics: particles

Particles behave as waves as well as particles (de Broglie). $p = h/\lambda$ so $\lambda = h/p$

Heisenburg uncertainty principle for particle waves $\Delta x \Delta p_x \geq \hbar/2$ as a fundamental limit to knowledge.

Bohr atom: electron orbits quantised angular momentum $L = n\hbar$. Gave quantised $E_n = -13.6/n^2$ eV

4 quantum mechanics

Schroedinger: wave equation to do wave-particle duality for matter. 1d, time dependent, Justified (not derived) from conservation of energy $E=KE+PE$.

$$E\Psi = \hbar\omega\Psi = (\hbar^2 k^2 / (2m) + U)\Psi$$
$$\frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U\Psi = i\hbar \frac{\partial \Psi}{\partial t} = E\Psi$$

$\Psi(x, t)$ is the wavefunction. This is NOT a physical observable. Probability to detect the particle at time t within dx of x is $|\Psi(x, t)|^2 dx = \Psi(x, t)^* \Psi(x, t) dx$.

The wavefunction is separable for time independent potential, $\Psi(x, t) = \psi(x)T(t)$, where $T(t) = e^{-iEt/\hbar}$ and $\psi(x)$ is given by the time independent Schroedinger equation

$$\frac{-\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

1 D infinite potential well. $U = 0$ for $0 < x < L$, $U = \infty$ elsewhere:

$$\frac{d^2 \psi}{dx^2} \psi(x) = -k^2 \psi(x) \quad \text{where } k^2 = 2mE/\hbar^2$$

solution is $\psi(x) = Ae^{ikx} + Be^{-ikx}$ (A is incident wave left to right, B is reflected wave, right to left). Boundary conditions give that these form a standing wave pattern with $\psi(x) = \sqrt{2/L} \sin n\pi x/L$ and $E_n = n^2\pi^2\hbar^2/(2mL^2)$ where $n = 1 \dots$

single states are stationary (standing waves) $\Psi(x, t) = \psi(x)T(t) = \psi_n(x)e^{-iE_n t/\hbar}$
so probability within dx of x at time t is

$$P(x, t)dx = \Psi(x, t)^*\Psi(x, t)dx = \psi_n^*(x)e^{iE_n t/\hbar}\psi_n(x)e^{-iE_n t/\hbar}dx = \psi_n^*(x)\psi_n(x)dx$$

Mixtures of different n will oscillate in time!

1 D finite potential: $U = 0$ for $0 < x < L$ and $U = U_0$ elsewhere. $E < U_0$ then wavefunction penetrates into the classically forbidden region $x > L$ with exponential decay

$$\frac{d^2\psi}{dx^2}\psi(x) = \alpha^2\psi(x) \quad \text{where } \alpha^2 = 2m(U_0 - E)/\hbar^2$$

$\psi(x) \propto e^{-\alpha(x-L)}$ and $E_n = n^2\pi^2\hbar^2/(2mL^2)$. If the potential barrier stops after some distance D then there is some probability that the electron is found on the other side of the barrier. Quantum tunnelling controls nuclear fusion in the sun, and alpha particle radioactive decays, and...

1 D harmonic oscillator $U = 1/2k'x^2$: $E_n = (n + 1/2)\hbar\omega$ where $\omega = \sqrt{k'/m}$ and $n = 0 \dots$. Zero point energy from Heisenberg!

Schrodinger: wave equation to do wave-particle duality for matter. 3d, time dependent, cartesian coordinates.

simplify for time independent potential then $\Psi(x, y, z, t) = \psi(x, y, z)T(t)$ where $T(t) = e^{-iEt/\hbar}$ as before. and for $U(x, y, z) = U_x(x) + U_y(y) + U_z(z)$ then it separates further into $\psi(x, y, z) = X(x)Y(y)Z(z)$ and there are three time independent Schrodinger equations for each of the functions X, Y, Z and three independent quantum numbers, n_x, n_y, n_z and three independent energies E_x, E_y, E_z . the total energy $E = E_x(n_x) + E_y(n_y) + E_z(n_z)$ and there may be multiple different ways to get the same energy (degeneracy), linked to symmetry

3D infinite potential well (isotropic) $0 < x < L, 0 < y < L, 0 < z < L$ and get

$$\psi(x, y, z) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z) = \sqrt{8/L^3} \sin n_x\pi x/L \sin n_y\pi y/L \sin n_z\pi z/L$$

and $E = \pi^2\hbar^2/(2mL^2)(n_x^2 + n_y^2 + n_z^2)$

3D harmonic potential (isotropic) $U = 1/2k'(x^2 + y^2 + z^2)$ and $E = (n_x + n_y + n_z + 3/2)\hbar\omega$

Schrodinger in spherical polars for hydrogen as $U(r) \neq U_x(x)+U_y(y)+U_z(z)$. but still separable. $\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi) = R(r)Y(\theta, \phi)$. The boundary conditions give 3 quantum numbers, n, l, m_l , where n (principle quantum number) sets the energy levels $E_n = -13.6/n^2$ eV, $l = 0, 1 \dots (n-1)$ (orbital angular momentum) and $m_l = -l \dots l$ (magnetic quantum number) in integer steps. The last two quantise angular momentum, with total magnitude squared $L^2 = (L_x^2 + L_y^2 + L_z^2) = l(l+1)\hbar^2$ and projection on the z -axis of $L_z = m_l\hbar$. We can't know L_y and L_z simultaneously as $\vec{L} = \vec{r} \times \vec{p}$ i.e. this would require that we knew position and momentum simultaneously in all directions which we cannot do because of Heisenburg. Label orbitals as n and letter denoting l . $l=0$:s, 1 :p, 2 :d, 3 :f etc. so 2s etc

The spatial part of the wavefunction is separable into radial wavefunctions $R_{nl}(r)$ and angular wavefunctions $Y_l^{m_l}(\theta, \phi)$. The probability of finding an electron within dr of r and $d\theta$ of θ and $d\phi$ of ϕ is $|\psi(r, \theta, \phi)|^2 dV$ BUT $dV = r^2 \sin\theta dr d\theta d\phi$. The probability of finding within dr of r is

$$\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} |\psi(r, \theta, \phi)|^2 r^2 \sin\theta dr d\theta d\phi$$

$$= r^2 |R_{nl}(r)|^2 dr \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} |Y_l^{m_l}(\theta, \phi)|^2 \sin\theta d\theta d\phi = r^2 |R_{nl}(r)|^2 dr$$

Electrons with orbital angular momentum form a current loop, so have a magnetic moment, so a response to a magnetic field. A magnetic field in the z -direction acts on the magnetic moment in the z direction and produces a potential $U = \mu_B m_l B_z$. So states with different m_l have a different response to the magnetic field, and their energy levels shift (Zeeman effect). Rather than looking at energy, we can SEE the effect via a deflection in a magnetic field with a gradient as $F_z = dU/dz$. if $dB_z/dz \neq 0$ then we get a force $\mu_B m_l dB_z/dz$ (Stern Gerlach apparatus)

But states with NO angular momentum also are shifted (anomolous Zeeman

effect), and show deflections. The deflection is quantised into 2 separate states. so we infer that electrons have SPIN, an intrinsic angular momentum, with total $S^2 = s(s + 1)\hbar^2$ and $S_z = m_s\hbar$ where $m_s = -s \dots s$ in integer steps. Since we have 2 separate spots, we have 2 separate values of m_s so $m_s = -1/2, 1/2$ and $s = 1/2$. This is INTRINSIC to all electrons (like charge).

Pauli exclusion principle - no 2 electrons can be in the same state (works for all spin 1/2 particles: fermions). So we need separate sets of quantum numbers n, l, m_l, m_s for each electron. this gives a zeroth order approximation for other elements. take nuclear charge Ze and assume energy levels unchanged from hydrogen at $-13.6Z^2/n^2$ eV and fill up the states. spectroscopic notation as before, but now with number of electrons in the shell e.g. $1s^22s^1$ for lithium. Outer electrons see Z_{eff} not entire charge Z as they are shielded by the inner filled shells (Debye shielding).

5 Quantum mechanics: nuclei

Z is number of protons, N is number of neutrons and $A=N+Z$ is number of nucleons. Mass is $\approx Au$ but not quite due to binding energy $E_B = (Nm_n + Zm_p - m)c^2$ where m is measured mass, and E_B/c^2 is called the mass defect. E_B has distinct peaks showing that there is structure to the energy levels in the nucleus. Nuclear potential is different for neutrons than protons as protons also feel electrostatic repulsion. But can be approximated by a 3D harmonic oscillator. In which case there are filled shells which have extra stability (like chemical stability for filled shell atomic orbitals for He, Ne, Ar etc). Filled shells are for $n_x = n_y = n_z = 1$ (x2 spin states). so 2 protons is stable, and 2 neutrons, and both is doubly stable. next is (1,1,2),(1,2,1) and (2,1,1) so 3 states (x2 spin states) is 6, so total of 8 protons or 8 neutrons and doubly stable with both.

radioactive decay: α decay - He particles. ${}^A_ZN \rightarrow {}^{A-4}Z - 2M + {}^4_2He$ where M is the element 2 down from N . Need mass of M to be bigger than combined mass of N and He.

β decay - electrons ${}^A_ZN \rightarrow {}^A_{Z+1}M + e^-$ where mass of N has to be bigger than

mass of M , and M is next element UP.