ELEMENTS OF QUANTUM MECHANICS - EXAMPLES CLASS 1 SOLUTION

i)

$$\int_0^L \Psi^*\Psi dx = 1 = N^2 \int_0^L \sin^2(\pi x/L)dx = N^2[\frac{x}{2} - \frac{\sin 2\pi x/L}{4\pi/L}]_0^L = N^2[\frac{L}{2} - \frac{L\sin 2\pi}{4\pi} - 0 + 0] = N^2\frac{L}{2}$$

hence $N = \sqrt{\frac{2}{L}}$ as required

$$P(0 < x < L/4) = \frac{2}{L}\int_0^{L/4} \sin^2(\pi x/L)dx = \frac{2}{L}[\frac{L}{8} - \frac{L\sin\pi/2}{4\pi} - 0 + 0] = \frac{2}{L}[\frac{L}{8} - \frac{L}{4\pi}] = 0.0908$$

ii)

$$< x^2 >= \frac{2}{L}\int_0^L x^2 \sin^2(\pi x/L)dx = \frac{2}{L}[-\frac{xL^2\cos(2\pi x/L)}{4\pi^2} - \frac{L^3(2\pi^2x^2/L^2 - 1)\sin(2\pi x/L)}{8\pi^3} + \frac{x^3}{6}]_0^L$$

$$= \frac{2}{L}[-\frac{L^3}{4\pi^2} - 0 + \frac{L^3}{6} - (-0 - 0 + 0)] = \frac{2}{L}(\frac{L^3}{6} - \frac{L^3}{4\pi^2}) = \frac{L^2}{3}(1 - \frac{3}{2\pi^2})$$

iii)

$$< p^2 >= \frac{-2\hbar^2}{L}\int_0^L \sin(\pi x/L)\frac{\partial^2[\sin(\pi x/L)]}{\partial x^2}dx$$

$$= \frac{-2\hbar^2}{L}\int_0^L \sin^2(\pi x/L)\frac{-\pi^2}{L^2}dx$$

$$= \frac{-2\hbar^2}{L}\frac{-\pi^2}{L^2}\frac{L}{2}$$

$$= \frac{\hbar^2\pi^2}{L^2}$$

hence $< T >=< p^2 > /2m = \frac{\hbar^2\pi^2}{2mL^2}$

iv) $< p >= 0$ its a standing wave so it has no momentum! and $< x >= L/2$ from symmetry. or just do the integrals

v)

$$(\Delta x)^2 =< x^2 > - < x >^2= \frac{L^2}{3}\Big(1 - \frac{3}{2\pi^2}\Big) - \frac{L^2}{4}$$

$$= 0.2826L^2 - 0.25L^2 = 0.0326L^2$$

$$\Delta x = 0.18L$$

$$(\Delta p)^2 = \hbar^2\pi^2/L^2 - 0$$

$$\Delta p = \hbar\pi/L$$

so $\Delta x\Delta p = 0.18\pi\hbar = 0.57\hbar$ This is bigger than $\hbar/2$, as required by the Heisenburg uncertainty principle.

vi)

$$< xp >= \frac{2}{L}\int_0^L \sin(\pi x/L)x - i\hbar\frac{\partial}{\partial x}[\sin(\pi x/L)]dx$$

$$= \frac{-2i\hbar}{L}\int_0^L x\sin(\pi x/L)\frac{\pi}{L}\cos(\pi x/L)dx$$

$$= \frac{-2i\hbar}{L}\frac{\pi}{L}\int_0^L x\sin(\pi x/L)\cos(\pi x/L)dx$$

$$= -\frac{2i\hbar\pi}{L^2}\frac{L^2}{8\pi^2}[\sin(2\pi x/L - 2\pi x/L\cos(2\pi x/L)]_0^L = -\frac{2i\hbar}{8\pi}[\sin(2\pi) - 2\pi\cos(2\pi) - 0 + 0]$$

$$= -\frac{2i\hbar}{8\pi}. - 2 = \frac{i\hbar}{2}$$

vii)

$$< px >= \frac{2}{L}\int_0^L \sin(\pi x/L) - i\hbar\frac{\partial}{\partial x}[x\sin(\pi x/L)dx]$$

$$= \frac{-2i\hbar}{L}\int_0^L \sin(\pi x/L)[\sin(\pi x/L) + x\frac{\pi}{L}\cos(\pi x/L)dx]$$

$$= \frac{-2i\hbar}{L}[\int_0^L \sin^2(\pi x/L)dx + \frac{\pi}{L}\int_0^L x\sin(\pi x/L)\cos(\pi x/L)dx$$

$$= \frac{-2i\hbar}{L}[\frac{L}{2} + \frac{\pi}{L}\frac{-L^2}{4\pi}]$$

$$= \frac{-2i\hbar}{L}\frac{L}{4} = \frac{-i\hbar}{2}$$

hence $< xp > - < px >= \frac{i\hbar}{2} - \frac{-i\hbar}{2} = i\hbar$ as required.