ELEMENTS OF QUANTUM MECHANICS - EXAMPLES CLASS 3 SOLUTION

Q1)
0,0,0 is ground state energy $(3/2)\hbar\omega$ degeneracy 1 (non-degenrate)
0,0,1, and 0,1,0 and 1,0,0 all have energies $(5/2)\hbar\omega$ so degeneracy 3
0,0,2 and 0,2,0 and 2,0,0 and 0,1,1 and 1,0,1 and 1,1,0 all have energies $(7/2)\hbar\omega$ so degeneracy 6
0,1,2 and 0,2,1 and 1,0,2 and 1,2,0 and 2,0,1 and 2,1,0 and 3,0,0 and 0,0,3 and 0,3,0, and 1,1,1 all have energies $(9/2)\hbar\omega$ so degeneracy 10

Q2)

1. i)

$$a_+\psi_0(x) = A(-ip + m\omega x)Ne^{-m\omega x^2/2\hbar} = AN\Big(-i.-i\hbar\frac{d}{dx} + m\omega x)e^{-m\omega x^2/2\hbar}$$

$$= AN\Big(-\hbar\frac{d}{dx}e^{-m\omega x^2/2\hbar} + m\omega x e^{-m\omega x^2/2\hbar}\Big)$$

$$= AN\Big(-\hbar\frac{-2xm\omega}{2\hbar}e^{-m\omega x^2/2\hbar} + m\omega x e^{-m\omega x^2/2\hbar}\Big)$$

$$= AN\Big(xm\omega + m\omega x\Big)e^{-m\omega x^2/2\hbar} = 2ANm\omega x e^{-m\omega x^2/2\hbar}$$

ii)

$$[a_-, a_+] = a_-a_+a_+ - a_- = A^2(ip+m\omega x)(-ip+m\omega x) - A^2(-ip+m\omega x)(ip+m\omega x)$$

$$= A^2[(ip.m\omega x + m\omega x. - ip) - (-ip.m\omega x + m\omega x.ip)]$$

$$= A^2 m\omega i(px - xp + px - xp) = A^2 m\omega i 2[p,x] = \frac{1}{2\hbar\omega m}2m\omega i(-i\hbar) = 1$$

or a better way is to keep it as commutators

$$= A^2([ip, -ip] + [ip, m\omega x] + [m\omega x, -ip] + [m\omega x, m\omega x])$$

$$= A^2(im\omega[p,x] - im\omega[x,p]) = -A^2 im\omega[x,p] = -2A^2 im\omega i\hbar = 1$$

iii) $a_+ + a_- = A(-ip + m\omega x + ip + m\omega x) = 2Am\omega x$ so

$$x = \frac{\sqrt{2\hbar m\omega}}{2m\omega}(a_+ + a_-) = \Big(\frac{\hbar}{2m\omega}\Big)^{1/2}(a_+ + a_-)$$

$$x^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)^2 = \frac{\hbar}{2m\omega}(a_+^2 + a_+a_- + a_-a_+ + a_-^2)$$

$$< V >= \frac{m\omega^2}{2}\int \psi_n^* x^2 \psi_n dx = \frac{m\omega^2}{2}\frac{\hbar}{2m\omega}\int \psi_n^*(a_+ + a_-)^2\psi_n dx$$

$$= \frac{\hbar\omega}{4} \int \psi_n^* (a_+^2 + a_+ a_- + a_- a_+ + a_-^2) \psi_n dx$$

$$= \frac{\hbar\omega}{4} \int \psi_n^* (A\psi_{n+2} + n\psi_n + (n+1)\psi_n + D\psi_{n-2}) dx$$

$$= \frac{\hbar\omega}{4} \int \psi_n^* (n\psi_n + (n+1)\psi_n) dx = \frac{\hbar\omega}{4}(2n+1) \int \psi_n^* \psi_n dx = \frac{\hbar\omega}{2}(n+1/2) = E_n/2$$