Example class 4 solution

(a) $[L^2, L_z]Y_{lm} = L^2(L_zY_{lm}) - L_z(L^2Y_{lm})$
$= L^2m\hbar Y_{lm} - L_zl(l+1)\hbar^2Y_{lm} = l(l+1)m\hbar^3Y_{lm} - m\hbar l(l+1)\hbar^2Y_{lm} = 0$
$30 = l(l+1)$ so $l = 5$ which means m takes any value $= -5, -4..0..4, 5$

(b) add to get $L_+ + L_- = 2L_x$ so $L_x = (L_+ + L_-)/2$
$< L_x >= \frac{1}{2}\int\int Y^*_{lm}L_+Y_{lm}\sin\theta d\theta d\phi + \frac{1}{2}\int\int Y^*_{lm}L_-Y_{lm}\sin\theta d\theta d\phi$
$= \frac{1}{2}A_{lm+}\int\int Y^*_{lm}Y_{lm+1}\sin\theta d\theta d\phi + \frac{1}{2}A_{lm-}\int\int Y^*_{lm}Y_{lm-1}\sin\theta d\theta d\phi = 0$
as normalisation is $\int\int Y_{lm}Y_{l'm'}\sin\theta d\theta d\phi = \delta(m-m')\delta(l-l')$

(c) $L_+L_-Y_{lm} \propto L + (Y_{l\ m-1}) \propto Y_{l\ m}$
this has the form of an eigenvalue/eigenfunction equation with $Y_{l\ m}$ as an eigenfunction.
$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + iL_yL_x - iL_xL_y + L_y^2$
$= L_x^2 + L_y^2 - i(L_xL_y - L_yL_x) = L_x^2 + L_y^2 - i[L_x, L_y] = L_x^2 + L_y^2 - ii\hbar L_z$
$= L^2 - L_z^2 + \hbar L_z$

$L_+L_-Y_{lm} = (L^2 - L_z^2 + \hbar L_z)Y_{lm} = \{l(l+1)\hbar^2 - m^2\hbar^2 + m\hbar^2\}Y_{lm}$
so eigenvalues $l(l+1)\hbar^2 - m^2\hbar^2 + m\hbar^2 = \hbar^2[l(l+1) - m(m-1)]$
$L_+L_-Y_{lm} = L_+(A_{lm-}Y_{l,m-1}) = \hbar\sqrt{l(l+1) - m(m-1)}L_+(Y_{l,m-1})$
Thi sis the tricky bit - remember you are now dealing with Y_{lm-1} NOT Y_{lm} so you have to find A_{lm-1} NOT A_{lm}
$= \hbar\sqrt{l(l+1) - m(m-1)}\hbar\sqrt{l(l+1) - (m-1)(m-1+1)}Y_{l,m-1}$
$= \hbar^2[l(l+1) - m(m-1)]Y_{lm}$ as required!

(d) $L_x\psi = q\hbar\psi = q\hbar(aY_{11} + bY_{10} + cY_{1-1})$
$L_-\psi = L_-(aY_{11} + bY_{10} + cY_{1-1}) = aA_{11}Y_{10} + bA_{10}Y_{1-1}$
$= a\hbar\sqrt{2}Y_{10} + b\hbar\sqrt{2}Y_{1-1}$
$L_+\psi = L_+(aY_{11} + bY_{10} + cY_{1-1}) = bA_{10}Y_{11} + cA_{10}Y_{10}$

$= \hbar(b\sqrt{2}Y_{11} + c\sqrt{2}Y_{10})$

Hence $L_x\psi = \frac{\hbar}{2}(b\sqrt{2}Y_{11} + (c+a)\sqrt{2}Y_{10} + b\sqrt{2}Y_{1-1}) = \frac{\hbar}{\sqrt{2}}(bY_{11} + (c+a)Y_{10} + bY_{1-1})$

equate coefficients $\frac{\hbar}{\sqrt{2}}b = \hbar aq$, $\frac{\hbar}{\sqrt{2}}(c+a) = \hbar bq$ and $\frac{\hbar}{\sqrt{2}}b = c\hbar q$

so for $q = 1$ we have $\frac{1}{\sqrt{2}}b = a$, $\frac{1}{\sqrt{2}}(c+a) = b$ and $\frac{1}{\sqrt{2}}b = c$

so $\psi = b(\frac{1}{\sqrt{2}}Y_{11} + Y_{10} + \frac{1}{\sqrt{2}}Y_{1-1})$

normalise $\psi = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}Y_{11} + Y_{10} + \frac{1}{\sqrt{2}}Y_{1-1})$

$= \frac{1}{2}Y_{11} + \frac{1}{\sqrt{2}}Y_{10} + \frac{1}{2}Y_{1-1}$

so for $q = -1$ we have $\frac{1}{\sqrt{2}}b = -a$, $\frac{1}{\sqrt{2}}(c+a) = -b$ and $\frac{1}{\sqrt{2}}b = -c$

so $\psi = b(-\frac{1}{\sqrt{2}}Y_{11} + Y_{10} - \frac{1}{\sqrt{2}}Y_{1-1})$

normalise $\psi = \frac{1}{\sqrt{2}}(-\frac{1}{\sqrt{2}}Y_{11} + Y_{10} - \frac{1}{\sqrt{2}}Y_{1-1}) = -\frac{1}{2}Y_{11} + \frac{1}{\sqrt{2}}Y_{10} - \frac{1}{2}Y_{1-1}$

and for $q = 0$ we have $\frac{1}{\sqrt{2}}b = 0$, $\frac{1}{\sqrt{2}}(c+a) = 0$ and $\frac{1}{\sqrt{2}}b = 0$ i.e. $b = 0$ and $c = -a$

so $\psi = a(Y_{11} - Y_{1-1})$

normalise $\psi = \frac{1}{\sqrt{2}}(Y_{11} - Y_{1-1})$