An electron of mass m_e is trapped in a 3D isotropic harmonic potential $V(x,y,z) = \frac{1}{2}m\omega^2(x^2+y^2+z^2)$ with unperturbed energy levels $E^0_{nx,ny,nz} = (n_x+n_y+n_z+3/2)\hbar\omega$.

(a) The system is perturbed by a potential $H' = \lambda x^2 yz$ where λ is a constant. Use non-degenerate perturbation theory to calculate the energy shift of the ground state $E^1_{0,0,0} =< \psi^0_{0,0,0}|H'\psi^0_{0,0,0} >$ where $\psi^0_{0,0,0} = (a/\pi)^{1/4}e^{-ax^2/2}e^{-ay^2/2}e^{-az^2/2}$ for $a = m_e\omega/\hbar$. [1 mark]

(b) The first excited state is triply degenerate with unperturbed wavefunctions

$$\psi^0_1 = \psi^0_{0,0,1} = Aze^{-ax^2/2}e^{-ay^2/2}e^{-az^2/2}$$
$$\psi^0_2 = \psi^0_{0,1,0} = Aye^{-ax^2/2}e^{-ay^2/2}e^{-az^2/2}$$
$$\psi^0_3 = \psi^0_{1,0,0} = Axe^{-ax^2/2}e^{-ay^2/2}e^{-az^2/2}$$

where $A = (2a)^{1/2}(a/\pi)^{3/4}$. Evaluate the matrix elements $W_{ij} =< \psi^0_i|H'\psi^0_j >$ for the perturbation in (a), where i, j take values $1, 2, 3$ denoting each wavefunction. [3 marks]

(c) Solve the resulting matrix equation for all possible values of E^1

$$\begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = E^1 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Is the level still degenerate? Write down the three wavefunctions $\chi = \alpha\psi_1 + \beta\psi_2 + \gamma\psi_3$ which correspond to each possible energy. [6 marks]

Useful Integrals

$$\int_{-\infty}^{+\infty} ue^{-\alpha u^2}du = 0, \qquad \int_{-\infty}^{+\infty} u^2e^{-\alpha u^2}du = \frac{1}{2}\left(\frac{\pi}{\alpha^3}\right)^{1/2}$$