(a) $E^1_{0,0,0} = \int\int\int Ne^{-ax^2/2}e^{-ay^2/2}e^{-az^2/2}\lambda x^2yzNe^{-ax^2/2}e^{-ay^2/2}e^{-az^2/2}dxdydz$

$= N^2\lambda\int\int\int x^2yze^{-ax^2}e^{-ay^2}e^{-az^2}dxdydz$

$= 0$ as $\int ye^{-ay^2}dy = 0$ (and same in z) [1 mark]

(b) $W_{11} = A^2\int\int\int z^2e^{-ax^2}e^{-ay^2}e^{-az^2}\lambda x^2yzdxdydz = 0$

$W_{22} = A^2\int\int\int y^2e^{-ax^2}e^{-ay^2}e^{-az^2}\lambda x^2yzdxdydz = 0$

$W_{33} = A^2\int\int\int x^2e^{-ax^2}e^{-ay^2}e^{-az^2}\lambda x^2yzdxdydz = 0$ [1 mark]

$W_{12} = A^2\int\int\int yze^{-ax^2}e^{-ay^2}e^{-az^2}\lambda x^2yzdxdydz$

$= A^2\lambda\int x^2e^{-ax^2}dx\int y^2e^{-ay^2}dx\int z^2e^{-az^2}dz$

$= A^2\lambda\frac{1}{2}\left(\frac{\pi}{a^3}\right)^{1/2}\frac{1}{2}\left(\frac{\pi}{a^3}\right)^{1/2}\frac{1}{2}\left(\frac{\pi}{a^3}\right)^{1/2}$

$= 2a\left(\frac{a}{\pi}\right)^{3/2}\lambda\frac{1}{8}\left(\frac{\pi}{a^3}\right)^{3/2} = \frac{\lambda}{4a^2}$ [1 mark]

$W_{13} = A^2\int\int\int xze^{-ax^2}e^{-ay^2}e^{-az^2}\lambda x^2yzdxdydz = 0$

$W_{23} = A^2\int\int\int yxe^{-ax^2}e^{-ay^2}e^{-az^2}\lambda x^2yzdxdydz = 0$

$W_{ij} = W^*_{ji}$ and all terms are real so $W_{ij} = W_{ji}$ [1 mark]

(c) let $b = \lambda/(4a^2)$ then the matrix is

$$\begin{pmatrix} -E^1 & b & 0 \\ b & -E^1 & 0 \\ 0 & 0 & -E^1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

[1 mark]

non trivial solution when determinant is zero so

$-E^1(E^1)^2 - b(-bE^1) = 0$ so $E^1[b^2 - (E^1)^2] = 0$ i.e. $E^1 = 0$ and $E^1 = \pm b$. [1 mark]

so the degeneracy is completely lifted. [1 mark]

for $E^1 = b$ then the matrix becomes

$$\begin{pmatrix} -b & b & 0 \\ b & -b & 0 \\ 0 & 0 & -b \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so $-b\alpha + b\beta = 0$ so $\alpha = \beta$ and $\gamma = 0$ so $\chi_1 = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ [1 mark]

for $E^1 = -b$ we have $b\alpha + b\beta = 0$ so $\alpha = -\beta$ and $\gamma = 0$ so $\chi_2 = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2)$ [1 mark]

fo $E^1 = 0$ the matrix gives all $\alpha, \beta, \gamma = 0$ so $\chi_3 = \psi_3$ as this is the only one which works for all values of E^1. [1 mark]