7.2 Eigenvectors of L^2

so now we know that $L_z\Phi_m = m\hbar\Phi_m$ where $\Phi_m = (2\pi)^{-1/2}e^{im\phi}$. But we also know that there has to be a common set of eigenfunctions which are BOTH eigenfunctions of L_z AND of L^2. We will call these $Y_{lm}(\theta, \phi)$. We already know that these have to be eigenfunctions of L_z so

$$L_z Y_{lm}(\theta\phi) = m\hbar Y(\theta\phi)$$

but these must also be eigenfunctions of L^2 so

$$L^2 Y_{lm}(\theta\phi) = l(l+1)\hbar^2 Y_{lm}(\theta\phi)$$

where again we have choosen to scale by $\hbar^2$ and the reason for calling the eigenvalue $l(l+1)$ will become clear soon!

We can see that $Y_{lm}(\theta\phi)$ must be separable into $\Theta_{lm}(\theta)\Phi_m(\phi)$ where Φ_m is as above and Θ can only be a function of θ and not ϕ as otherwise it would be changed by $L_z = -i\hbar\frac{\partial}{\partial\phi}$ and then this wouldn't be an eigenfunction of both of them.

so solving

$$L^2 Y_{lm}(\theta\phi) = l(l+1)\hbar^2 Y_{lm}(\theta\phi)$$

$$L^2\Theta(\theta)\Phi_m(\phi) = l(l+1)\hbar^2\Theta(\theta)\Phi_m(\phi)$$

$$-\hbar^2\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Big)\Theta(\theta)\Phi_m(\phi) = l(l+1)\hbar^2\Theta(\theta)\Phi_m(\phi)$$

$$\frac{\Phi}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta}) + \frac{\Theta}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = -l(l+1)\Theta(\theta)\Phi_m(\phi)$$

eigenfunctions of L_z had

$$\frac{\partial\Phi_m}{\partial\phi} = im\Phi_m$$

$$\frac{\partial^2\Phi_m}{\partial\phi^2} = im\frac{\partial\Phi_m}{\partial\phi} = -m^2\Phi_m$$

substitute in

$$\frac{\Phi}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Theta}{\partial\theta}) - \frac{\Theta}{\sin^2\theta}m^2\Phi = -l(l+1)\Theta(\theta)\Phi_m(\phi)$$

divide out Φ to get just a function of θ so $\partial/\partial\theta \rightarrow d/d\theta$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) - \frac{\Theta}{\sin^2\theta}m^2 = -l(l+1)\Theta(\theta)$$

and rearrange

$$\Big(\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d}{d\theta}) + l(l+1) - \frac{m^2}{\sin^2\theta}\Big)\Theta_{lm} = 0$$

the equation is ugly but well known mathematically. When $m = 0$ the solutions are Legendre polynomials $P_l(\cos\theta)$, where the higest order polynomial term is of order l where $l = 0, 1, 2\dots$ and is called the orbital angular momentum quantum number some examples are

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2 = \frac{1}{2}(3\cos^2\theta - 1)$$
$$P_3 = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

so like the hermite polynomials, the pattern is that even l only has even powers of $\cos\theta$, odd l only has odd powers of $\cos\theta$.

for $m \neq 0$ the solutions are given by the associated Legendre polynomials which are related to the $|m|^{th}$ derivative of P_l. But since P_l is a polynomial of degree l, then its l+1 derivative will vanish. so for a fixed value of l, $|m|$ can only take the values $0\dots l$, so the allowed values of m for a given l are

$$m = -l, -l+1, -l+2\dots 0, 1, 2\dots(l-1), l$$

so there are 2l+1 values of m for every l.

Then we normalise these to get our solutions for $\Theta_{lm}(\theta)$. the normalisation integral is only over θ so $\int \Theta^*_{lm}(\theta)\Theta_{lm}(\theta)\sin\theta d\theta$. then we get

$$\Theta_{lm} = (-1)^m\Big(\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\Big)^{1/2}P_l^m(\cos\theta) \quad m \geq 0$$
$$\Theta_{lm} = (-1)^m\Theta_{l|m|} \quad m < 0$$

7.3 Spherical harmonics $Y_{lm}(\theta\phi)$

the eigenvectors common to L^2 and L_z are given by

$$Y_{lm} = \Theta_{lm}\Phi_m = (-1)^m \Big(\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\Big)^{1/2} P_l^m(\cos\theta) e^{im\phi} \quad m \geq 0$$

$$Y_{lm} = (-1)^m Y^*_{l,-m} \quad m < 0$$

with allowed values $m = 0, \pm 1, \pm 2 \ldots \pm l$.

we can now see the limit on values which m can take more physically as

$$< L^2 >=< L_x^2 + L_y^2 + L_z^2 >=< L_x^2 > + < L_y^2 > + < L_z^2 >$$

since these operators are all hermitian then these are all real so

$$< L^2 > \; \geq \; < L_z^2 >$$

$$\int\int Y^*_{lm} L^2 Y_{lm} \sin\theta d\theta d\phi \geq \int\int Y^*_{lm} L_z^2 Y_{lm} \sin\theta d\theta d\phi$$

$$\int\int Y^*_{lm} l(l+1)\hbar^2 Y_{lm} \sin\theta d\theta d\phi \geq \int\int Y^*_{lm} L_z(m\hbar Y_{lm}) \sin\theta d\theta d\phi$$

$$l(l+1)\hbar^2 \geq \int\int Y^*_{lm}(m^2\hbar^2 Y_{lm}) \sin\theta d\theta d\phi$$

$$l(l+1) \geq m^2$$

its obviously true for $m = l$, and not true for $m = l+1$ as the rhs is l^2+2l+1 which is bigger than the lhs of $l^2 + l$. so this limits the values of $|m| \leq l$

The fact that we have determined L_z means that L_x and L_y cannot be simultaneously determined - if we measure them we will get a value which is quantised at $0, \pm\hbar, \pm 2\hbar$ etc but the actual value is uncertain. However, we can explicitally evaluate their averages, $< L_x >$ and $< L_y >$. And these both turn out to be zero. so although the particular value of L_x and L_y cannot be predicted, their average value can.

So how do we visualise a system where we have L^2 and L_z with quantised values, but where $< L_x >$ and $< L_y >= 0$? we can do this with a classical vector model. The angular momentum vector of magnitude $\sqrt{l(l+1)}\hbar$ precesses around the z axis, so the z component is always fixed at $m\hbar$. Because of the precession, $< L_x >$ and $< L_y >$ vanish.

so thats the agnular momentum done. what about the wavefunctions themselves? we can plots these Y_l^m out - A polar plot represents the magnitude (absolute value) of the function as the length of a line centered at the origin, and it represents the values of the angles θ, ϕ by the direction of the line.

but more useful than wavefunctions of course is probability distributions.

The probability of finding the particle within volume $dV = dA = \sin\theta d\theta d\phi$ of position θ, ϕ is $dP = |Y_{lm}|^2 \sin\theta d\theta d\phi$ so if we want the probability density in ϕ then we have to integrate over all theta $dP = D(\phi)d\phi = \int_{\theta=0}^{\pi} |Y_{lm}|^2 \sin\theta d\theta d\phi$ conversely, if we wanted the probability density in θ we'd do $dP = D(\theta)d\theta = \int_0^{2\pi} |Y_{lm}|^2 d\phi \sin\theta d\phi$ similarly, if we want the probabilty per unit area on a sphere, then its $dPdA = |Y_{lm}|^2 dA$ so

$$D(A) = Y_{lm}^* Y_{lm} = \Theta_{lm}^* \Phi_m^* \Theta_{lm} \Phi_m = (2\pi)^{-1} \Theta_{lm}^* \Theta_{lm}$$

The ϕ dependence drops out, so the probability of finding the particle is independent of ϕ so we can plot these more easily. This gives a polar diagram, showing the dependence of the probability on θ. A polar plot represents the magnitude (absolute value) of the function as the length of a line centered at the origin, and it represents the values of the angle θ by the direction of the line.