9 More on angular momentum

We saw from Hydrogen that each level n had n^2 degenerate levels, n^2 states with different wavefunctions but the same energy. but they actually don't. thats becasue we haven't put in spin yet! then we can have fine structure from spin-orbit coupling. and relativistic effects also give us fine structure too. and then there is also hyperfine structure from coupling to the nuclear spin.

so, lets do spin. in classical mechanics then there are two kinds of angular momentum, orbital and spin. this division is a bit arbitrary as the two are really the same thing.

but in quantum mechanics the distinction is fundamental. orbital angular momentum of the electron about the nucleus is described by spherical harmonics. but the electron carries anolther angular momentum which has nothing to do with its position in space so cannot be described by $r\theta\phi$. intead, an electron has intrinsic angular momentum as well as the extrinsic orbital angular momentum.

we're going to describe it similary to the orbital angular momentum though, so first a quick review of L, and its extension to ladder operators. and then we will use the same notation to define spin!

9.1 Angular momentum: Smart way!

$[L_x, L_y] = i\hbar L_z$, and likewise $[L_y, L_z] = i\hbar L_x$ and $[L_y, L_z] = i\hbar L_x$. These cannot be measured together.

we also stated (without proof) that $[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$. so any ONE of the components of L can be measured together with its total magnitude. and we normally choose L_z as it has the simplest mathematical form in spherical polar coordinates.

we solved this the brute force and ignorance way and found $L^2 Y_{lm} = l(l +$

1)$\hbar^2 Y_{lm}$ and $L_z Y_{lm} = m\hbar Y_{lm}$ where m takes integer values from $-l \ldots l$.

But suppose now we didn't know that. suppose instead we start at the beginning and know only that we have some operator L with components L_x, L_y, L_z who obey the commutation relations $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$, $[L_y, L_z] = i\hbar L_x$ and $[L^2, L_z] = 0$. This tells us that L^2 and L_z share some common set of eigenvectors $f_{\lambda\mu}$ so $L^2 f_{\lambda\mu} = \lambda\hbar^2 f_{\lambda\mu}$ and $L_z f_{\lambda\mu} = \mu\hbar f_{\lambda\mu}$

9.2 ladder operators

we can also form an operator $L\pm = L_x \pm iL_y$. lets take L_+ to be definite.

$$[L^2, L_+] = [L^2, L_x + iL_y] = [L^2, L_x] + i[L^2, L_y] = 0$$

so this is something we can measure along with L^2, so it shares common eigenfunctions $f_{\lambda\mu}$ too.

$$[L^2, L_+]f_{\lambda\mu} = L^2(L_+ f_{\lambda\mu}) - L_+(L^2 f_{\lambda\mu}) = L^2(L_+ f_{\lambda\mu}) - \lambda(L_+ f_{\lambda\mu}) = 0$$

so $L_+ f_{\lambda\mu}$ is an eigenvector of L^2 and its the one with eigenvalue $\lambda\hbar^2$.

lets see what happens with L_z.

$$[L_z, L_+] = [L_z, L_x] + i[L_z, L_y] = i\hbar L_y + i. - i\hbar L_x = \hbar L_x + i\hbar L_y = \hbar L_+$$

so these DON'T have common eigenfunctions. but what does it do?

$$[L_z, L_+]f_{\lambda\mu} = \hbar L_+ f_{\lambda\mu}$$

$$L_z(L_+ f_{\lambda\mu}) - L_+(L_z f_{\lambda\mu}) = \hbar L_+ f_{\lambda\mu}$$

$$L_z(L_+ f_{\lambda\mu}) - L_+ \mu\hbar f_{\lambda\mu} = \hbar L_+ f_{\lambda\mu}$$

$$L_z(L_+ f_{\lambda\mu}) = L_+(\mu+1)\hbar f_{\lambda\mu} = (\mu+1)\hbar(L_+ f_{\lambda\mu})$$

so $L_+ f_{\lambda\mu}$ is an eigenfunction of L_z but with an eigenvalue $(\mu+1)\hbar$ not $\mu\hbar$ which is what we started with. so the operator L_+ is a raising operator - we can raise μ by one each time. $L_+ f_{\lambda\mu} = N f_{\lambda\mu+1}$

and similarly we have a lowering operator $L_z(L_- f_{\lambda\mu}) = (\mu-1)\hbar L_- f_{\lambda\mu}$

so for a given value of λ there is a ladder of states, with each rung of the ladder separated by one unit of $\hbar$ in L_z. To ascend the ladder we use L_+, to decend we use L_-. But this can't go on forever, there is a top rung - where L_z^2 exceeds the total angular momentum L^2, where we can't raise it any more. so at the top value of $\mu = t$, $L_+ f_{\lambda t} = 0$

now, lets do a bit of sleight of hand. If we raise the index and then lower it again we get

$$L_- L_+ = (L_x - iL_y)(L_x + iL_y) = (L_x^2 + L_y^2 - iL_y L_x + iL_x L_y)$$

$$= L^2 - L_z^2 + i[L_x, L_y] = L^2 - L_z^2 + i.i\hbar L_z = L^2 - L_z^2 - \hbar L_z$$

so if we do this on the top rung we know that $L_+ f_{\lambda t} = 0$ so we have

$$L_-(L_+ f_{\lambda t}) = L^2 f_{\lambda t} - L_z^2 f_{\lambda t} - \hbar L_z f_{\lambda t} = 0$$

$$0 = \lambda\hbar^2 f_{\lambda t} - t^2\hbar^2 f_{\lambda t} - t\hbar^2 f_{\lambda t}$$

or $\lambda = t(t+1)$. so now we can see why we chose L^2 eigenvalues to be $l(l+1)$ rather than some constant λ as doing it this way we explicitly set the eigenvalue of L^2 to set the range we need in the eigenvalues of L_z

we can do the whole thing again and get that the bottom value of $\mu = b$ where we need to lower the index, and then raise it i.e. $L_+(L_- f_{\lambda b}) = 0$ and then we get $\lambda = b(b-1)$

but this is all for the same value of λ so $b(b-1) = t(t+1)$ so either $b = t+1$ which is ridiculous since this would make the bottom rung higher than the top rung, or $b = -t$. so μ runs from $-t$ to t in integer steps where $\lambda = t(t+1)$.

The number of steps N needs to be an integer. So -t+N=t so $t = N/2$.

This tells us something MORE than our first way with polymonials. This tells us μ can be an integer OR HALF INTEGER. Yet our polynomical approach only gave us an integer. There is something about real orbital angular momentum which forced integer values on us so we had

$$L^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm} \quad l = 0, 1, \ldots$$

$$L_z Y_{lm} = m\hbar Y_{lm} \quad -l, -l+1 ..., l-1, l$$

But if all we had was the commuters, we could define some general angular momentum which could have half integer values of μ.

9.3 General angular momentum: J

Any vector J is defined to be an angular momentum if its componets J_x, J_y, J_z satisfy the commutation relations $[J_x, J_y] = i\hbar J_z$, $[J_y, J_z] = i\hbar J_x$ and $[J_z, J_x] = i\hbar J_y$ and have a total $J^2 = J_x^2 + J_y^2 + J_z^2$ which commutes with all its components so that $[J^2, J_x] = [J^2, L_y] = [J^2, L_z] = 0$. Then there are common eigenfunctions of J^2 and J_z called f_{jm_j}, which are defined to have eigenvalues values $j(j+1)\hbar^2$ and $m_j\hbar$, respectively. So $J^2 f_{j,m_j} = j(j+1)\hbar^2 f_{j,m_j}$ and $J_z f_{j,m_j} = m_j \hbar f_{j,m_j}$

These also have ladder operators $J_\pm = J_x \pm iJ_y$, such that J_+ raises m_j by unity, while J_- lowers it by unity. so

$$J_z J_+ = (m_j + 1)\hbar J_+ \quad J_z J_- = (m_j - 1)\hbar J_-$$

this cannot go on indefinitely as we know that $J^2 = J_x^2 + J_y^2 + J_z^2 \geq J_z^2$ so $j(j+1) \geq m_j^2$ Hence there is a top value of $m_j = m_t$ and there is also a bottom value $m_j = m_b$. since we are going up and down in integer steps then $m_t - m_b = N$ where $N = 0, 1, 2.....$ $J_-(J_+ f_{j,m_t}) = 0$ implies $j(j+1) = m_t(m_t+1)$ and $J_+(J_- f_{j,m_b}) = 0$ implies $j(j+1) = m_b(m_b-1)$ which shows that $m_b = -m_t$ so $m_t = j$ and $-j + N = j$ where N is an integer so $2j = N$ so $j = 0, 1/2, 1, 3/2...$

$$J^2 f_{jm_j} = j(j+1)\hbar^2 f_{jm_j} \quad j = 0, 1/2, 1, 3/2...$$
$$J_z f_{jm_j} = j\hbar f_{jm_j} \qquad m_j = -j, -j+1, -j+2...0....j-1, j$$

A GENERAL angular momentum can have integer OR HALF INTEGER values of j, with m_j, running up to $\pm j$

9.4 Spin

so we define S as an angular momentum spin operator, with S^2 eigenvalues $s(s+1)\hbar^2$ and S_z eigenvalues $m_s\hbar$. THEIR EIGENFUNCTIONS ARE NOT spherical harmonics! they are not functions of $\theta\phi$ at all.

every elementary particle has a specific and immutable value of s which is its intrinsic spin. fermions have spin $s = 1/2$ so m_s can take values $\pm 1/2$. this is in sharp contrast to the orbital angular momentum l which can take any allowed value $l = 0...n-1$ and is NOT fixed - it can change as the system is perturbed.

electrons are femions so they can exist in only one of two eigenstates of spin, spin up $m_s = +1/2$, eigenvector χ_+ or spin down $m_s = -1/2$, eigenvector χ_-.

there is good experimental evidence for this - the Stern-Gerlach experiment. They took silver atoms - Z=47. This has a single outer electron in the n=5, l=0, m=0 level. with $l = 0$ then the electron has zero angular momentum and therefore produces no current loop so should not interact with an external

magnetic field. Stern and Gerlack took their beam of silver atoms through an inhomogeneous B field. The field separated the beam into two distinct parts!! if the electron had an intrinsic magnetic dipole then it will experience a force proportional to the field gradient since the two "poles" will be subject to different fields. this would give a continuous smear at the detectors if the dipole could be oriented in any direction. so to split into two requires that there are only two directions allowed!

lets see this in more detail:

For an electron moving in an orbit of radius r with speed v

$$\begin{aligned} A &= \pi r^2 \\ I &= -\frac{ev}{2\pi r} \end{aligned}$$

and

$$\mu_l = IA = -\frac{evr}{2} = -\frac{e}{2m_e} m_e v r = -\frac{e}{2m_e} L \tag{1}$$

where L is the orbital angular momentum of the electron. In vector form

$$\boldsymbol{\mu_l} = -\frac{e}{2m_e}\mathbf{L} = -\frac{\mu_B}{\hbar}\mathbf{L} \tag{2}$$

where $\mu_B = e\hbar/2m_e$ is called the Bohr magneton, and is a natural unit of microscopic magnetic moment, with the value 9.27×10^{-24} J T^{-1} or 5.79×10^{-5} eV T^{-1}.

Therefore it is reasonable to identify $-(\mu_B/\hbar)\mathbf{L}$ as the quantum mechanical magnetic moment operator associated with orbital angular momentum. It follows that the operator for the z-component of the magnetic moment is

$$(\mu_l)_z = -\frac{\mu_B}{\hbar} L_z$$

An ideal measurement of the quantity $(\mu_l)_z$ must yield one of the eigenvalues of the corresponding operator. Hence, for a hydrogen atom with orbital angular momentum quantum number l, the possible values of the quantity $(\mu_l)_z$ are $-m_l\mu_B$, where m_l is one of the numbers from $-l$ and $+l$ in steps of unity.

Now consider a hydrogen atom in a z-directed magnetic field B_z. A classical model would give the associated magnetic potential energy as

$$-(\mu_l)_z B_z = m\mu_b B_z$$

In classical magnetism, a uniform magnetic field creates a torque, but no translational force, on an object possessing a magnetic moment. However, a translational force can be produced by the application of a spatially varying field - as force is the derivative of potential $F = -dV/dz$ so we might expect the quantum mechanical force on a hydrogen atom to be

$$F_z = (\mu_l)_z \frac{dB_z}{dz} \tag{3}$$

where $(\mu_l)_z$ is equal to one of the discrete values $-m_l\mu_B$.

but if we have the outer electron in the $l = 0$ state then $m = 0$ so there should be no effect!

but the beam IS split into two in the Stern-Gerlach experiment, showing that there IS a magnetic moment which is NOT associated with orbital angular momentum, and can take 2 potential values rather than the continuum of values you might expect with a randomly aligned spin dipole. This motivates us to assicated a magnetic moment $\boldsymbol{\mu_s}$ with the spin angular momentum $\mathbf{S}$. we'll assume

$$\boldsymbol{\mu_s} = -g_s \frac{e}{2m_e}\mathbf{S} = -g_s \frac{\mu_B}{\hbar}\mathbf{S} \tag{4}$$

where g_s is called the spin g-factor. so then the force is

$$F_z = (\mu_s)_z \frac{\partial B_z}{\partial z} = -g_s m_s \mu_B \frac{\partial B_z}{\partial z} \tag{5}$$

Now we know that two lines are seen in the experiment, which implies that F_z has two possible values for each hydrogen atom. That in turn suggests that m_s can have two values. If we assume that the allowed values of m_s must range from $-s$ to $+s$ in steps of unity, in analogy with the relation between m_l and l in the orbital case, then we must take $s = 1/2$ and $m_s = \pm 1/2$. Since the spin is assumed to be an intrinsic property of the electron, we take the picture to be valid for all electrons, and not just those in hydrogen atoms. Because $s = 1/2$ we refer to the electron as a spin-1/2 particle which can exist in the states $m_s = 1/2$ or $S_z = +\hbar/2$, called spin up, and $m_s = -1/2$ or $S_z = -\hbar/2$, called spin down.

Using $m_s = \pm 1/2$ and comparing with the force seen in experiment we find $g_s = 2$. So eq. 4 that

$$(\mu_s)_z = -g_s m_s \mu_B = \mp \mu_B \tag{6}$$