(a) $\int \psi^* \psi dx = 1$ [S:1 mark]

$$A^2 \int_0^1 x^2(1-x)^2 dx = A^2 \int_0^1 x^2(1-2x+x^2)dx$$

$$= A^2 \int_0^1 x^2 - 2x^3 + x^4 dx = A^2 \Big[\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{x^5}{5}\Big]_0^1$$

$$= A^2(1/3 - 1/2 + 1/5) = A^2/30$$

so $A^2/30 = 1$ hence $A = \sqrt{30}$ [U:1 mark]

$p = -i\hbar d/dx$ [S:1 mark]

$$< p >= \int \psi^* . - i\hbar \frac{d\psi}{dx} dx = -30i\hbar \int_0^1 (x-x^2)\frac{d}{dx}(x-x^2)dx$$

$$= -30i\hbar \int_0^1 (x-x^2)(1-2x)dx = -30i\hbar \int_0^1 x - x^2 - 2x^2 + 2x^3 dx$$

$$= -30i\hbar[x^2/2 - 3x^3/3 + 2x^4/4]_0^1 = -30i\hbar[1/2 - 1 + 1/2] = 0$$

[U:1 mark]

(b) $$< xp >= \int \psi^* x\Big(-i\hbar\frac{d\psi}{dx}\Big)dx = -30i\hbar \int_0^1 (x-x^2)x\frac{d}{dx}(x-x^2)dx$$

$$= -30i\hbar \int_0^1 (x^2-x^3)(1-2x)dx = -30i\hbar \int_0^1 x^2 - x^3 - 2x^3 + 2x^4 dx$$

$$= -30i\hbar[x^3/3 - 3x^4/4 + 2x^5/5]_0^1 = -30i\hbar(1/3 - 3/4 + 2/5) = i\hbar/2$$

[U:2 marks]

$$< px >= -i\hbar \int \psi^* \frac{d(x\psi)}{dx} dx = -30i\hbar \int_0^1 (x-x^2)\frac{d}{dx}x(x-x^2)dx$$

$$= -30i\hbar \int_0^1 (x-x^2)(2x-3x^2)dx = -30i\hbar \int_0^1 2x^2 - 2x^3 - 3x^3 + 3x^4 dx$$

$$= -30i\hbar[2x^3/3 - 5x^4/4 + 3x^5/5]_0^1 = -30i\hbar(2/3 - 5/4 + 3/5) = -i\hbar/2$$

so $< xp > - < px >= i\hbar/2 - -i\hbar/2 = i\hbar$ [U:2 marks]

(c) $\Psi(x,t) = \psi_n(x)e^{-iE_nt/\hbar}$ [S:2 marks]

$$< x >= \int \Psi^*(x,t)x\Psi(x,t)dx = \int \psi_n^*(x)e^{iE_nt/\hbar}x\psi_n(x)e^{-iE_nt/\hbar}dx$$

$$= \int \psi_n^*(x)x\psi_n(x)dx = \int |\psi_n(x)|^2 xdx$$

this is not time dependent - eigenfunctions are stationary states[S:2 marks]

(d) $\psi = \frac{1}{\sqrt{18}}(\psi_{100} + 3\psi_{200} + 2\psi_{211} + 2\psi_{321})$

prob $n = 1$ is $1/18$

prob $n = 2$ is $9/18 + 4/18 = 13/18$.

prob $n = 3$ is $4/18$ [U:1 mark]

$< E >= P_1E_1 + P_2E_2 + P_3E_3$ [U:1 mark]

$= -13.6(1/18.1 + 13/18.1/4 + 4/18.1/9) = -13.6 * 0.261 = -3.54$ eV [U:1 mark]

No, an ideal measurement will always give one of the eigenvalues. and this is not one of them. [U:1 mark]

(e) Probablity that found with $r_0 < r < r_0 + dr$ is

$$\int_{r_0}^{r_0+dr} \int_0^{\pi} \int_0^{2\pi} \psi^*\psi r^2 \sin\theta dr d\theta d\phi = 4\pi(\pi a^3)^{-1} \int_{r_0}^{r_0+dr} r^2 e^{-2r/a} dr$$

$$= \frac{4}{a^3} r^2 e^{-2r/a} dr$$

[S:2 marks]

so radial probability density is $4/a^3 r^2 e^{-2r/a}$. this has a maximum when

$$\frac{d}{dr}(\frac{4}{a^3} r^2 e^{-2r/a}) = 0$$

$$2re^{-2r/a} + r^2(-2/a)e^{-2r/a} = 0 \quad \text{so } 2r - 2r^2/a = 0 \quad \text{so } r = a$$

[S:2 marks]

f) J^2 has eigenvalues $j(j+1)\hbar^2$, L^2 has eigenvalues $l(l+1)\hbar^2$ and S^2 has eigenvalues $s(s+1)\hbar^2$ [S:2 marks]

$J^2 = (\underline{L}+\underline{S}).(\underline{L}+\underline{S}) = L^2 + 2\underline{L}.\underline{S} + S^2$

hence $\underline{L}.\underline{S} = \frac{1}{2}(J^2 - L^2 - S^2) = \frac{\hbar^2}{2}[j(j+1) - l(l+1) - s(s+1)]$[S:2 marks]

(g) $$\frac{E_{nj} - E_n}{E_n} = \frac{\alpha^2}{n^2}\Big(\frac{n}{j+1/2} - \frac{3}{4}\Big)$$

[U:1 mark]

$n=2$ has $l=0$ and 1. $j = l \pm 1/2$ for $l>0$ or $1/2$ for $l=0$. So $l=0$ has $j=1/2$. and $l=1$ can have $j=1/2$ (as before) or $3/2$. [S:1 mark]

$$j = 1/2 \quad n = 2 \quad \frac{\alpha^2}{4}\Big(\frac{2}{1/2+1/2} - \frac{3}{4}\Big) = 1.66 \times 10^{-5}$$

[U:1 mark]

$$j = 3/2, \quad n = 2 \quad \frac{\alpha^2}{4}\Big(\frac{2}{3/2+1/2} - \frac{3}{4}\Big) = 3.33 \times 10^{-6}$$

[U:1 mark]

(h) $$\begin{pmatrix} 1 & \kappa \\ \kappa & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E^1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} 1-E^1 & \kappa \\ \kappa & 1-E^1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

[U:1 mark]

only non-trivial solution when the determinant is zero. [U:1 mark]

$(1-E^1)^2 - \kappa^2 = 0$ so $1 - E^1 = \pm\kappa$ or $E^1 = 1 \pm \kappa$. [U:2 marks]

(a) $$< T >= -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\psi^*\frac{d^2}{dx^2}\psi dx$$

[S:1 mark]

$$= -\frac{\hbar^2}{2m}\Big(\frac{\beta^2}{\pi}\Big)^{1/2}\int_{-\infty}^{+\infty}e^{-\beta^2x^2/2}\frac{d}{dx}\frac{d}{dx}e^{-\beta^2x^2/2}dx$$

[U:1 mark]

$$= -\frac{\hbar^2}{2m}\Big(\frac{\beta^2}{\pi}\Big)^{1/2}\int_{-\infty}^{+\infty}e^{-\beta^2x^2/2}\frac{d}{dx}(-\beta^2xe^{-\beta^2x^2/2})dx$$

$$= \frac{\hbar^2}{2m}\frac{\beta^3}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\beta^2x^2/2}\frac{d}{dx}(xe^{-\beta^2x^2/2})dx$$

$$= \frac{\hbar^2}{2m}\frac{\beta^3}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\beta^2x^2/2}(e^{-\beta^2x^2/2} + x. - \beta^2.xe^{-\beta^2x^2/2})dx$$

[U:2 marks]

$$= \frac{\hbar^2}{2m}\frac{\beta^3}{\sqrt{\pi}}\Big(\int_{-\infty}^{+\infty}e^{-\beta^2x^2}dx - \beta^2\int_{-\infty}^{+\infty}x^2e^{-\beta^2x^2}dx\Big)$$

$$= \frac{\hbar^2}{2m}\frac{\beta^3}{\sqrt{\pi}}\Big(\frac{\sqrt{\pi}}{\beta} - \beta^2\frac{\sqrt{\pi}}{2\beta^3}\Big) = \frac{\hbar^2}{2m}\beta^2(1-1/2) = \frac{\hbar^2\beta^2}{4m}$$

[U:2 marks]

(b) $$< V >= \frac{\beta}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\beta^2x^2/2}\frac{m\omega^2x^2}{2}e^{-\beta^2x^2/2}dx$$

[U:2 marks]

$$= \frac{\beta}{\sqrt{\pi}}\frac{m\omega^2}{2}\int_{-\infty}^{+\infty}x^2e^{-\beta^2x^2}dx$$

$$= \frac{m\omega^2\beta}{2\sqrt{\pi}}\frac{\sqrt{\pi}}{2\beta^3} = \frac{m\omega^2}{4\beta^2}$$

[U:2 marks]

(c) $$< E > = < T > + < V > = \frac{\hbar^2\beta^2}{4m} + \frac{m\omega^2}{4\beta^2}$$

[U:2 marks]

(d) minimum when $dE/d\beta = 0$

$$\frac{dE}{d\beta} = 2\frac{\hbar^2\beta}{4m} - 2\frac{m\omega^2}{4\beta^3} = 0$$

$$\hbar^2\beta^4 - m^2\omega^2 = 0 \quad \text{so } \beta = \sqrt{\frac{m\omega}{\hbar}}$$

[U:2 marks]

$$< E > = \frac{\hbar^2 m\omega}{4m\hbar} + \frac{m\omega^2\hbar}{4m\omega} = \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}$$

[U:2 marks]

(e) Eigenfunction if $H\psi = E\psi$. [S:1 mark]

$$H\psi = \Big(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2x^2}{2}\Big)\Big(\frac{\beta^2}{\pi}\Big)^{1/4} e^{-\beta^2x^2/2}$$

the $d^2\psi/dx^2$ term was calculated in (a) above. So

$$= \Big(-\frac{\hbar^2}{2m}(-\beta^2 + \beta^4x^2) + \frac{m\omega^2x^2}{2}\Big)\psi$$

$$= -\frac{\hbar^2}{2m}\Big(-\beta^2 + \beta^4x^2 - \frac{m\omega^2x^2}{2}\frac{2m}{\hbar^2}\Big)\psi$$

[U:1 mark]

This is only an eigenfunction where the term in brackets does not depend on x, so that the energy is a constant. this requires

$$\beta^4 = \frac{m\omega^2}{2}\frac{2m}{\hbar^2} = m^2\omega^2/\hbar^2 \quad \text{so } \beta = \sqrt{m\omega/\hbar}$$

[U:2 marks]

(a) $$\int (\psi_n^0)^* H^0 \psi_n^1 dx + \int (\psi_n^0)^* H' \psi_n^0 = \int (\psi_n^0)^* E_n^0 \psi_n^1 dx + \int (\psi_n^0)^* E_n^1 \psi_n^0 dx$$

$$= E_n^0 \int (\psi_n^0)^* \psi_n^1 dx + E_n^1$$

[S:2 marks]

Hermitian $\int \psi_n^*(H\psi_m)dx = \int (H\psi_n)^* \psi_m dV$ [S:2 marks]

$$\int (H^0\psi_n^0)^* \psi_n^1 dx + \int (\psi_n^0)^* H' \psi_n^0 = E_n^0 \int (\psi_n^0)^* \psi_n^1 dx + E_n^1$$

$$\int (E_n^0\psi_n^0)^* \psi_n^1 dx + \int (\psi_n^0)^* H' \psi_n^0 = E_n^0 \int (\psi_n^0)^* \psi_n^1 dx + E_n^1$$

since $E_n^0 = E_n^{0*}$ (its real) then the first terms on each side of the equation cancel and

$$\int (\psi_n^0)^* H' \psi_n^0 = E_n^1$$

[S:2 marks]

(b) $$E_n^1 = 2/L \int \sin(n\pi x/L)^2 a\delta(x - L/3)dx$$

$$= \frac{2a}{L} \sin^2(n\pi/3)$$

[U:2 marks]

$n = 1$ $E_1^1 = 2a/L \sin^2(\pi/3) = (2a/L)(3/4) = 3a/(2L)$ [U:1 mark]

$n = 2$ $E_2^1 = 2a/L \sin^2(2\pi/3) = 3a/(2L)$ [U:1 mark]

$n = 3$ $E_3^1 = 2a/L \sin^2(3\pi/3) = 0$ [U:1 mark]

$$c_{12} = \frac{\int \psi_2^{0*} a\delta(x - L/3)\psi_1^0 dx}{E_1^0 - E_2^0}$$

$$= \frac{2mL^2}{(1-4)\pi^2\hbar^2}\frac{2}{L}\int \sin(2\pi x/L) a\delta(x-L/3)\sin(\pi x/L)dx$$

$$= -\frac{4mLa}{3\pi^2\hbar^2}\sin(2\pi/3)\sin(\pi/3) = -\frac{4mLa}{3\pi^2\hbar^2}\frac{3}{4} = -\frac{mLa}{\pi^2\hbar^2}$$

[U:2 marks]

$$c_{13} = \frac{\int \psi_3^{0*} a\delta(x-L/3)\psi_1^0 dx}{E_1^0 - E_3^0}$$

$$= \frac{2mL^2}{(1-9)\pi^2\hbar^2}\frac{2}{L}\int \sin(3\pi x/L) a\delta(x-L/3)\sin(\pi x/L)dx$$

$$= -\frac{4mLa}{8\pi^2\hbar^2}\sin(3\pi/3)\sin(\pi/3) = 0$$

[U:1 mark]

$$\psi_1^1 \approx c_{12}\psi_2^0 + c_{13}\psi_3^0 + \ldots = -\sqrt{\frac{2}{L}}\frac{mLa}{\pi^2\hbar^2}\sin(2\pi x/L)$$

[U:2 marks]

$$E_1^2 = -2/L\int \sin(\pi x/L) a\delta(x-L/3) \times \frac{mLa}{\pi^2\hbar^2}\sin(2\pi x/L)dx$$

[U:2 marks]

$$= -\frac{2a}{L}\frac{mLa}{\pi^2\hbar^2}\sin(\pi/3)\sin(2\pi/3)$$

$$= -\frac{2a}{L}\frac{mLa}{\pi^2\hbar^2}\frac{3}{4} = -\frac{3a^2m}{2\pi^2\hbar^2}$$

[U:2 marks]