Vorticity in the Early Universe

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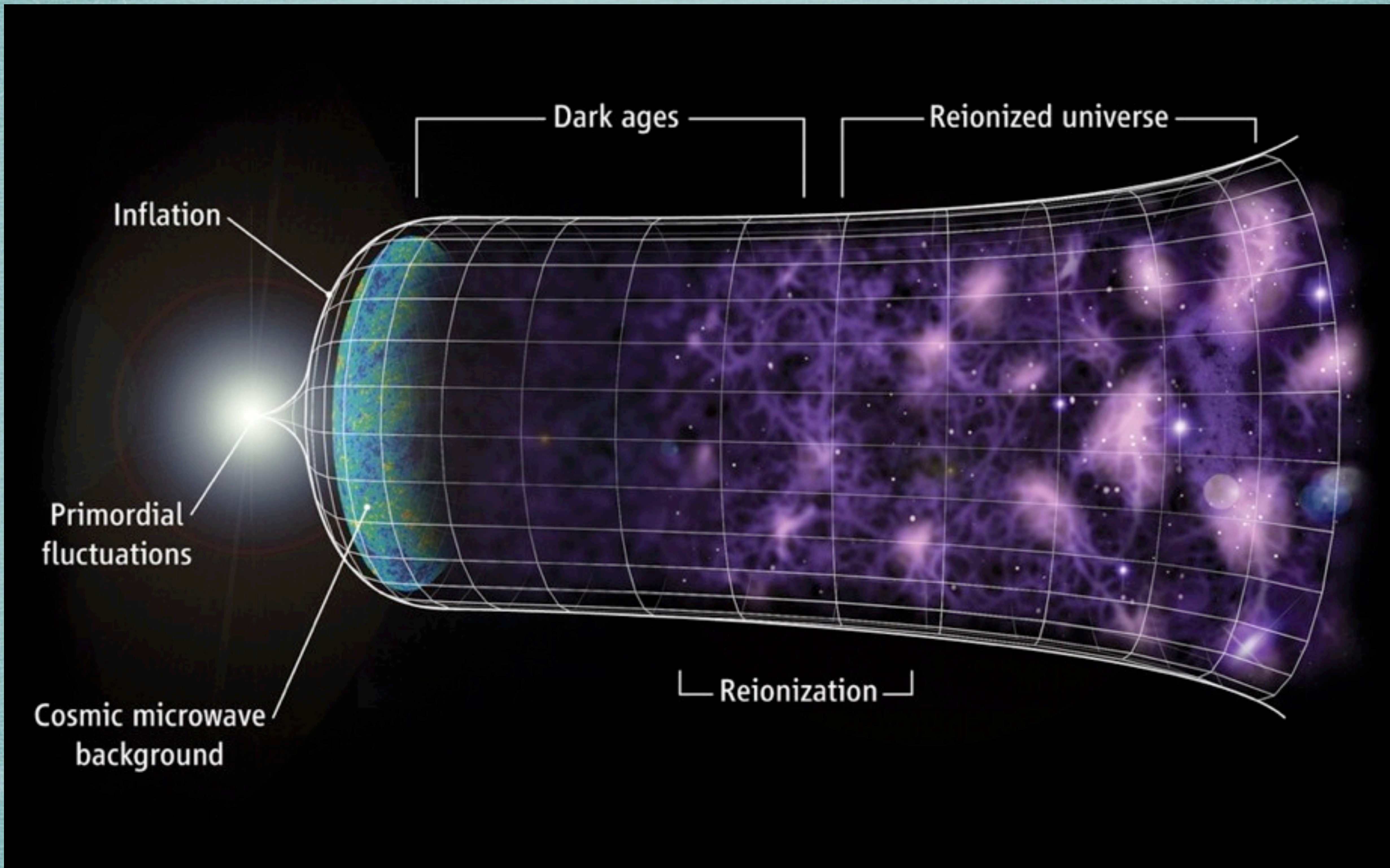
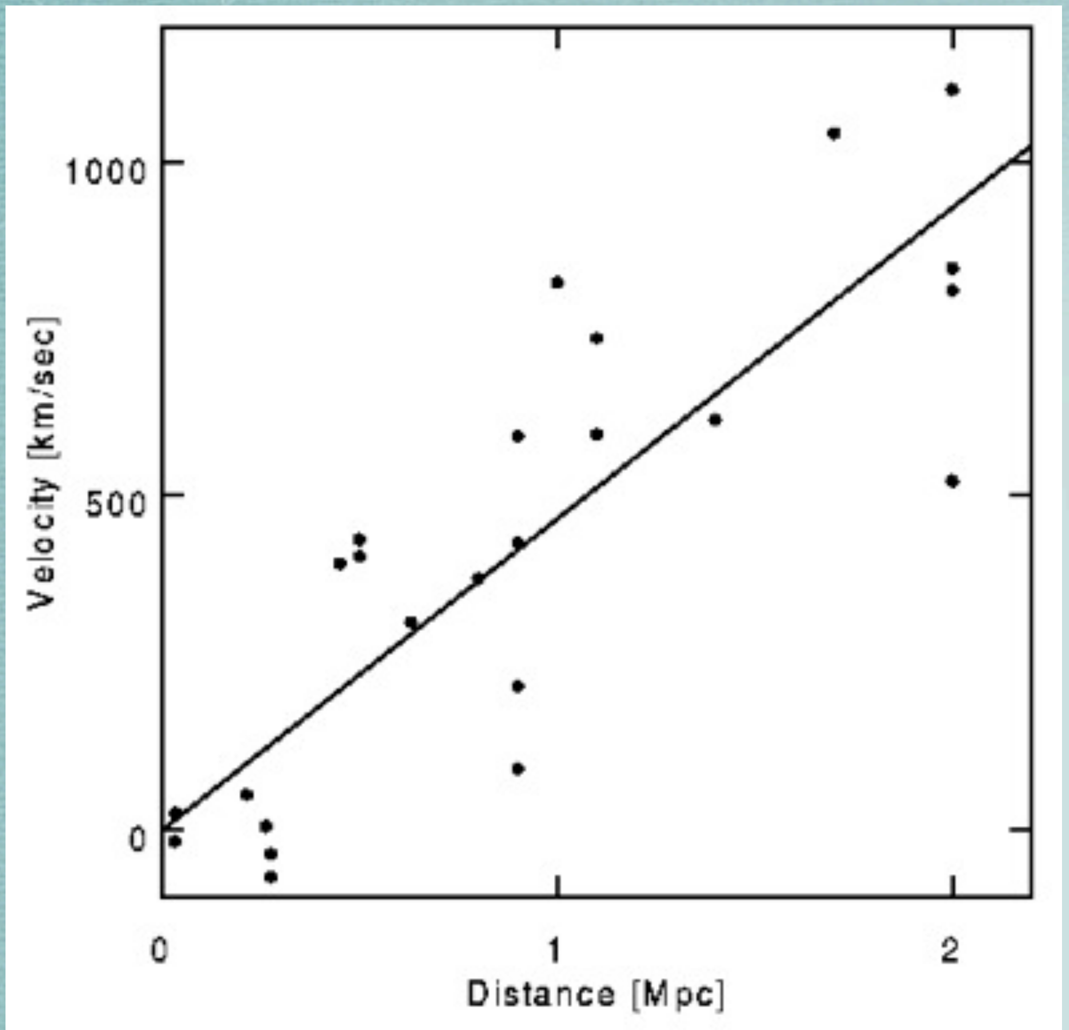
Outline

- Introduction and motivation
- Standard big-bang cosmology
- Modelling inhomogeneities
  - Newtonian perturbation theory
  - Relativistic perturbation theory
- Application: vorticity beyond linear order
- Summary and conclusions

Evolution of the Universe

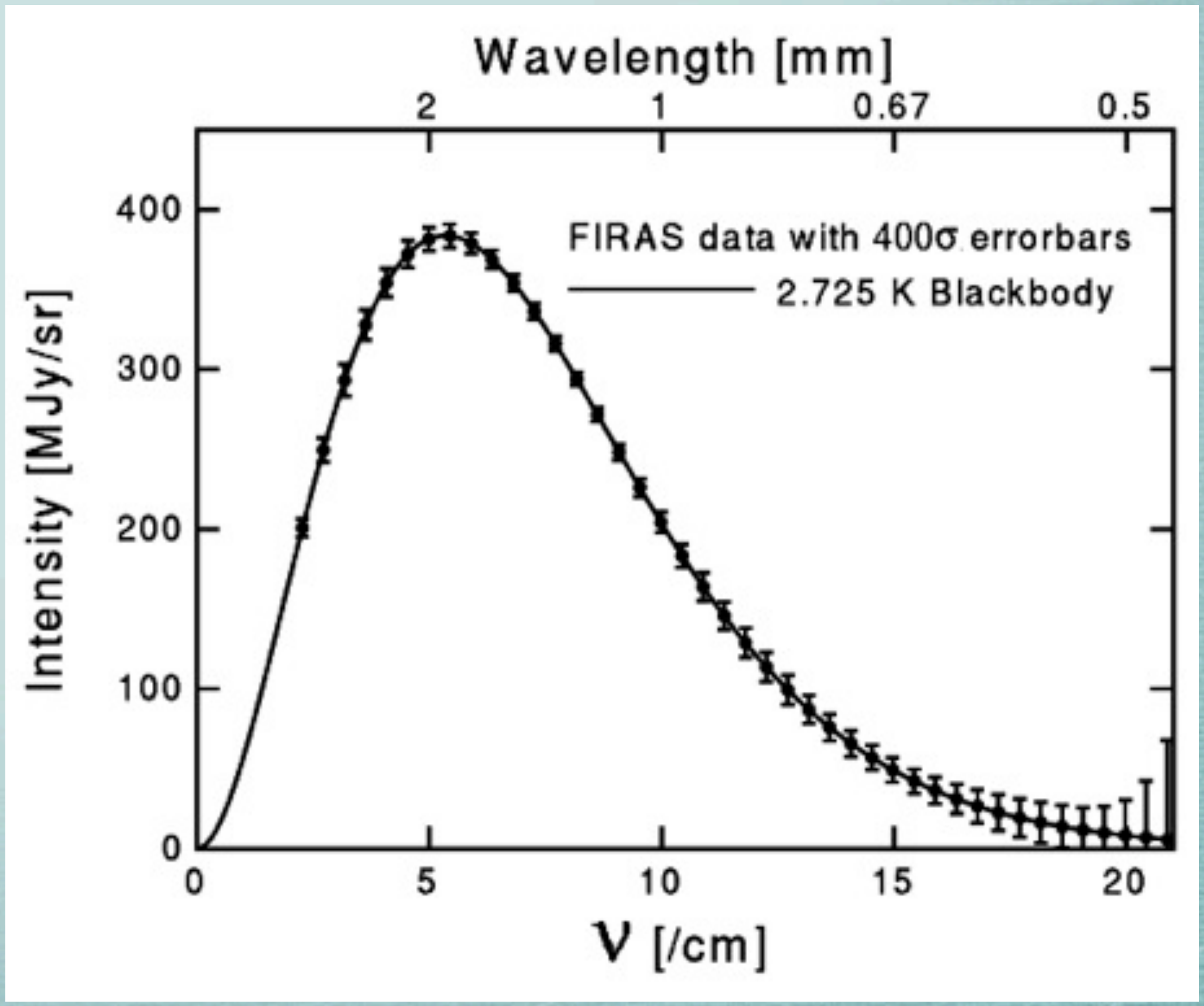


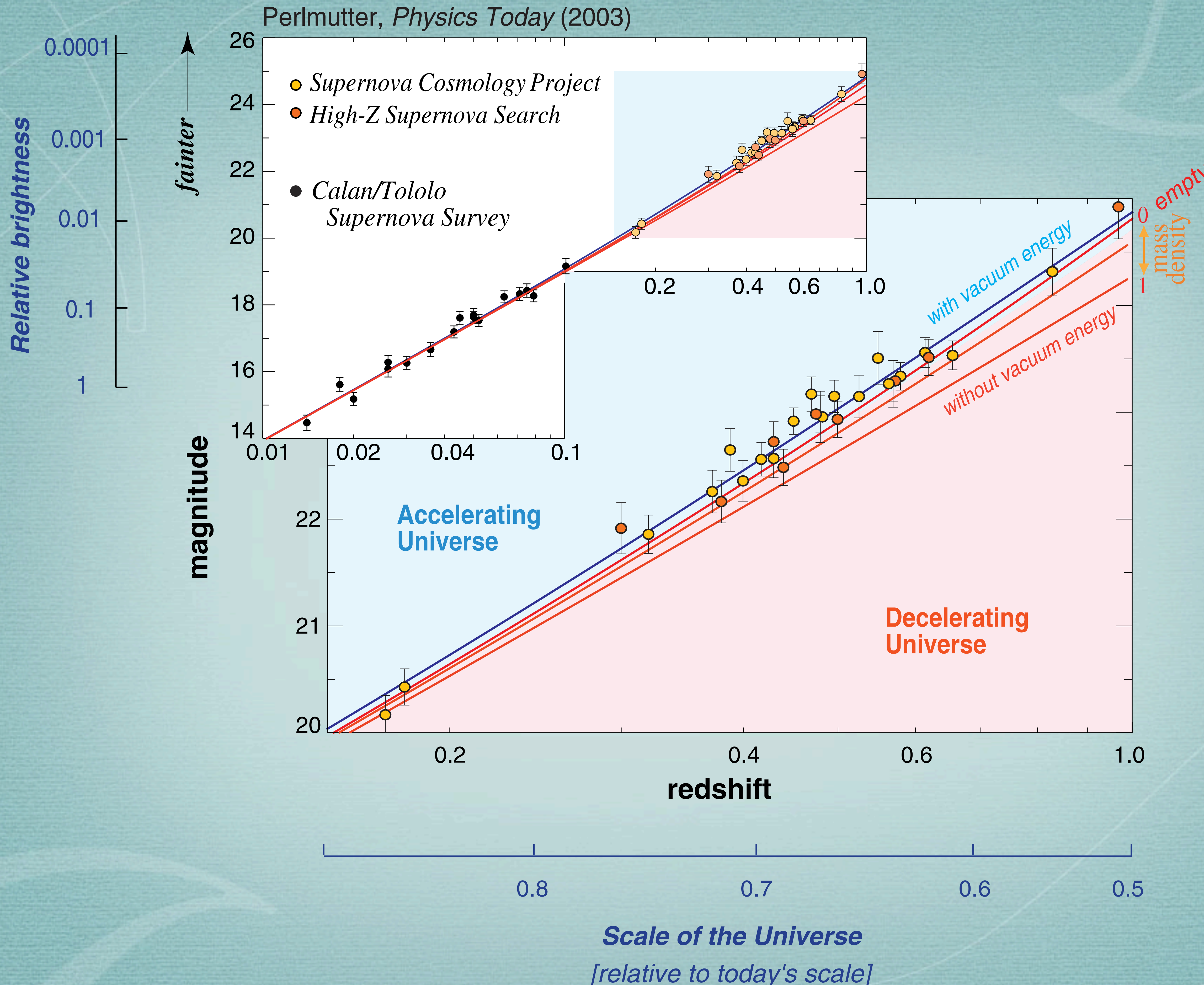
Observations



Hubble 1929 - first evidence for universe expansion

CMB as measured by COBE - perfect blackbody





Perlmutter, *Physics Today* (2003)
Supernova Cosmology Project
High-Z Supernova Search
Calan/Tololo Supernova Survey
Relative brightness
0.0001
0.001
0.01
0.1
1
fainter
magnitude
26
24
22
20
18
16
14
0.01
0.02
0.04
0.1
0.2
0.4
0.6
1.0
with vacuum energy
without vacuum energy
empty
0
1
mass density
Accelerating Universe
Decelerating Universe
22
21
20
redshift
0.8
0.7
0.6
0.5
Scale of the Universe
[relative to today's scale]

Friedmann models

* Homogeneous and isotropic solution to GR
* Scale factor, $a(t)$, obeys

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

* Energy conservation

$$\dot{\rho} = -3H(\rho + P)$$

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$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

spatial curvature
k=+1: open
k=0: infinite/flat
k=-1: closed

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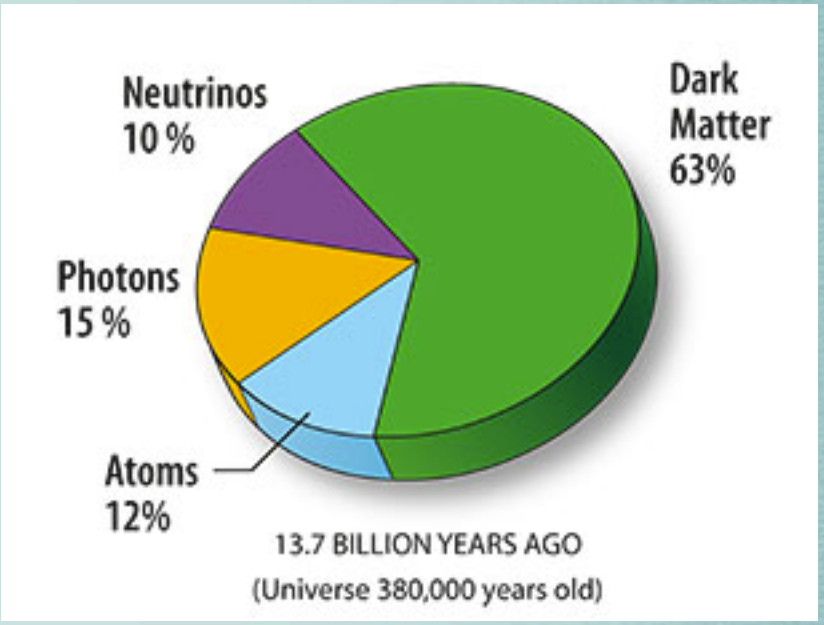
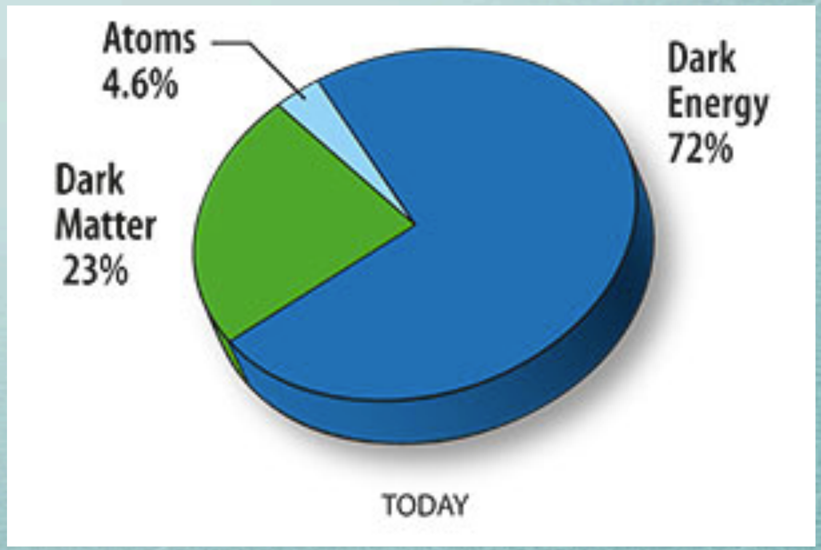
radiation, CDM, baryonic matter, dark energy...

$$\Omega_i = \frac{\rho_i}{\rho_{\rm crit}} = \frac{8\pi G}{3H^2}\rho_i$$

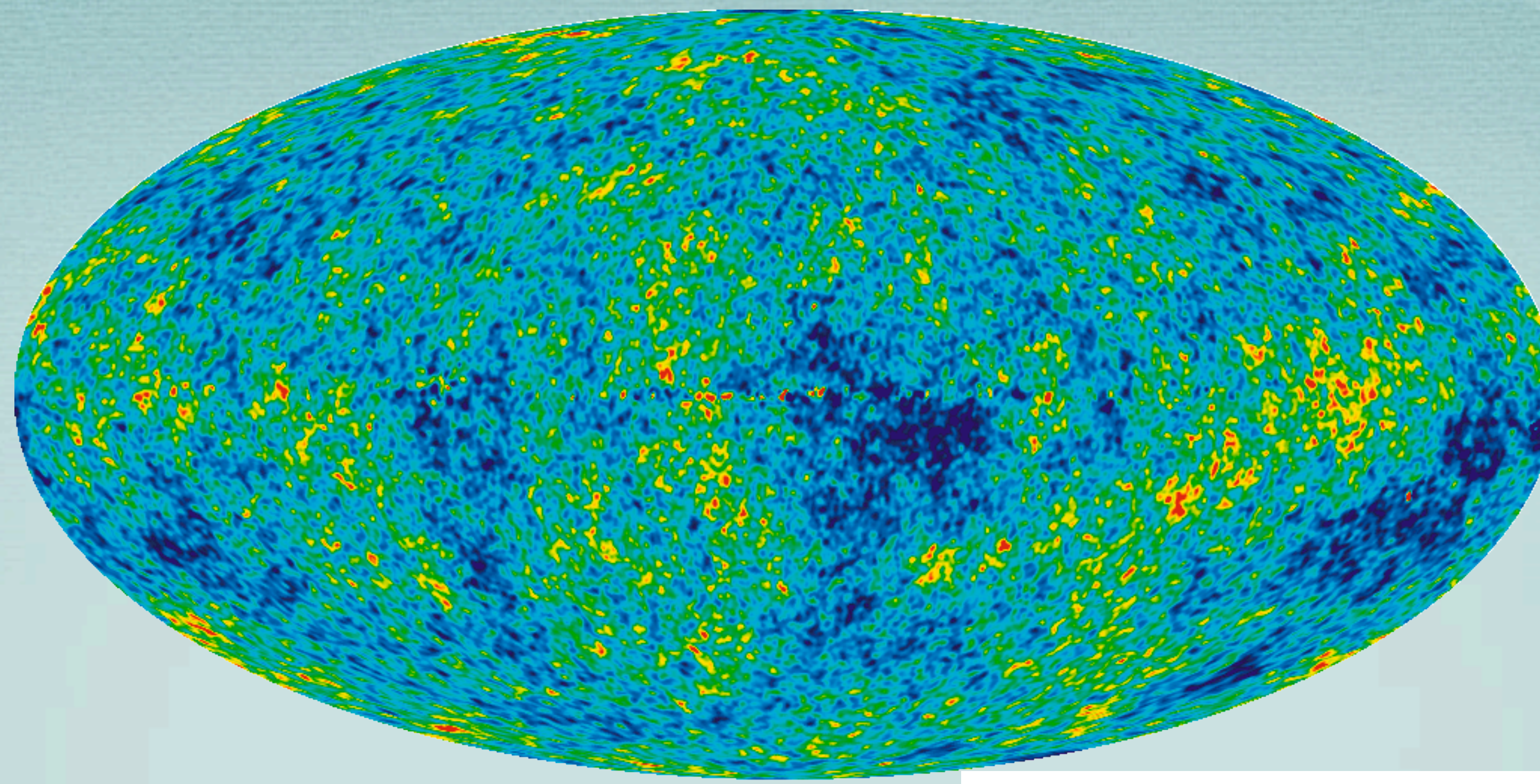
$$\Omega_k = -\frac{k}{Ha^2}$$

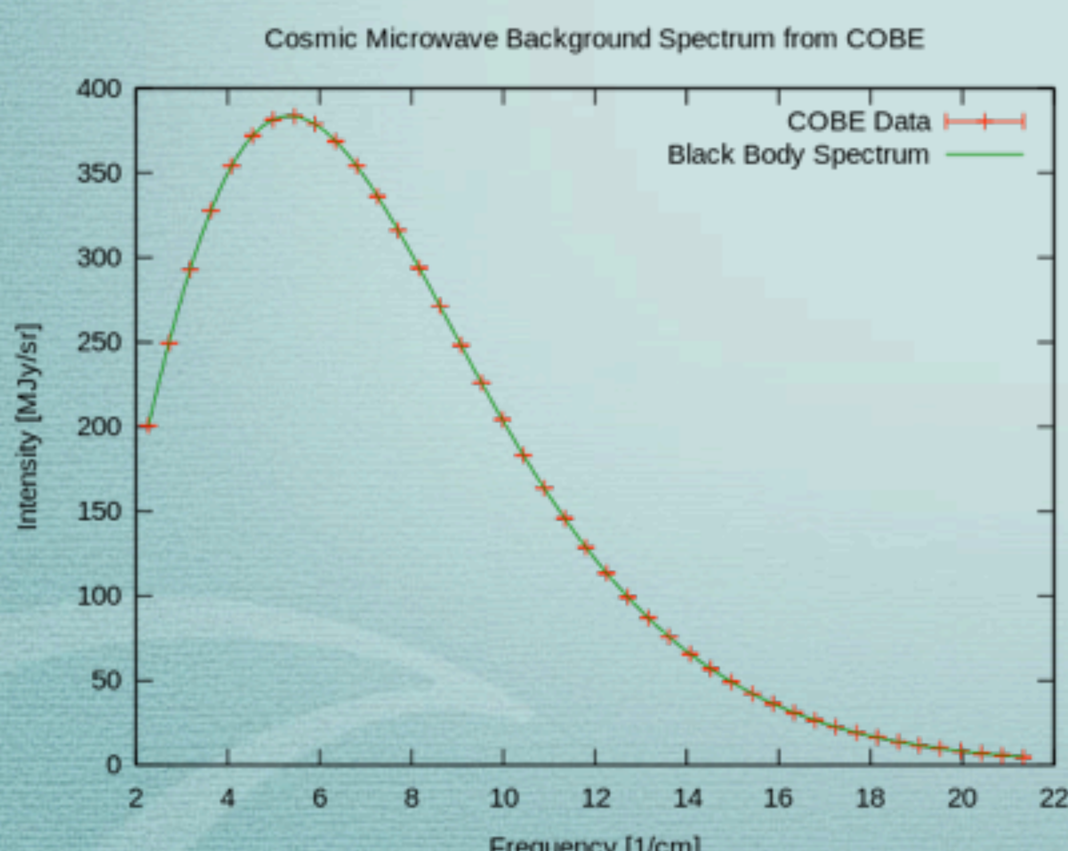
$$\Omega = \sum_i \Omega_i = 1 - \Omega_k$$

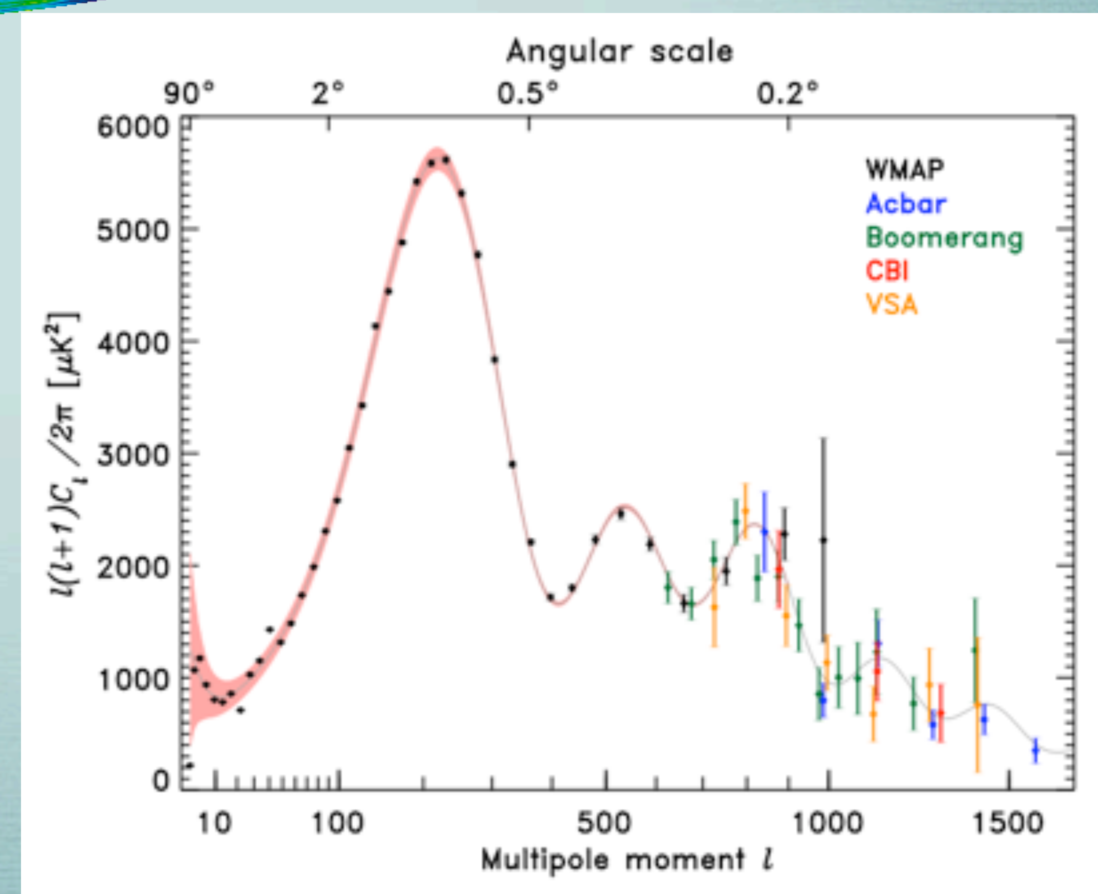




Microwave background







Modelling inhomogeneities

- Friedmann is an approximation: there exists structure (galaxies, stars, etc..), and CMB anisotropies
- Consider perturbations about a homogeneous 'background' solution
- e.g. write energy density as

$$\rho(\vec{x},t) = \bar{\rho}(t)\Big(1+\delta(\vec{x},t)\Big)$$

- newtonian mechanics...

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inhomogeneous perturbation

- newtonian mechanics...

Newtonian cosmology

- Newtonian perturbation theory:

  energy density: $\rho(\vec{x},t) = \bar{\rho}(t)\Big(1+\delta(\vec{x},t)\Big)$

  velocity: $\vec{v}(\vec{x},t)$, Newtonian potential: $\Phi(\vec{x},t)$

- Fluid evolution equations

$$\dot{\delta} + \vec{\nabla}\cdot\Big[(1+\delta)\vec{v}\Big] = 0$$

$$\dot{\vec{v}} + H\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}\Phi - \frac{\vec{\nabla}P}{\bar{\rho}(1+\delta)}$$

- Poisson equation

$$\nabla^2\Phi = 4\pi G\bar{\rho}a^2\delta$$

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* Linearised fluid equations

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* Poisson equation

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* Alternatively, writing $\delta P = c_{\rm s}^2 \delta\rho$, obtain

$$\frac{\partial^2 \delta}{\partial t^2} + 2H\frac{\partial \delta}{\partial t} = 4\pi G\bar{\rho}\delta + c_{\rm s}^2\nabla^2\delta$$

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Hubble drag: suppresses growth of perturbations

Pressure term

Gravitational term: perturbations grow via gravitational instability

Relativistic inhomogeneities

* General relativity governs dynamics of the universe
* Must use relativity to describe regions of high density, fluids moving an appreciable fraction of c, or large scales
* Einstein's field equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

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Einstein tensor, function of the metric tensor, describes geometry

energy momentum tensor, describes matter

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Cosmological perturbations

* How to proceed?
  - Fully inhomogeneous solution (*extremely* difficult in principle; impossible in practice?)
  - Similar to Newtonian case: expand around a homogeneous solution - **Cosmological Perturbation Theory**
* Inhomogeneous perturbations to

matter, e.g., energy density $\rho(\vec{x},t) = \bar{\rho}(t)\Big(1 + \delta(\vec{x},t)\Big)$

geometry: metric tensor $g_{\mu\nu}(\vec{x},t) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t)$

$$g_{\mu\nu}(\vec{x},t) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t)$$

* FLRW metric:

$$[g^{(0)}_{\mu\nu}] = \begin{bmatrix} 1 & 0 \\ 0 & a^2(t)\delta_{ij} \end{bmatrix}$$

- homogeneous & isotropic
- take flat spatial space in agreement with observations

$$g_{\mu\nu}(\vec{x},t) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t)$$

* Perturbed FLRW metric:

  two independent scalars, e.g.

$$[\delta g_{\mu\nu}] = \begin{bmatrix} -2\Phi(\vec{x},t) & 0 \\ 0 & a^2(t)2\Psi(\vec{x},t)\delta_{ij} \end{bmatrix}$$

  or

$$[\delta g_{\mu\nu}] = \begin{bmatrix} -2\phi(\vec{x},t) & a(t)B_{,i}(\vec{x},t) \\ a(t)B_{,i}(\vec{x},t) & 0 \end{bmatrix}$$

* Different 'gauges' - can choose to work with different variables depending on the problem at hand

$$g_{\mu\nu}(\vec{x},t) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t)$$

- Perturbed FLRW metric:

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Newtonian gauge

$$[\delta g_{\mu\nu}] = \begin{bmatrix} -2\Phi(\vec{x},t) & 0 \\ 0 & a^2(t)2\Psi(\vec{x},t)\delta_{ij} \end{bmatrix}$$

or

Uniform curvature gauge

$$[\delta g_{\mu\nu}] = \begin{bmatrix} -2\phi(\vec{x},t) & a(t)B_{,i}(\vec{x},t) \\ a(t)B_{,i}(\vec{x},t) & 0 \end{bmatrix}$$

- Different 'gauges' - can choose to work with different variables depending on the problem at hand

Governing equations

* Fluid equations

$$\delta' + (1+w)(\nabla^2 v - 3\Psi') = 3\mathcal{H}(w - c_{\rm s}^2)\delta$$

$$v' + \mathcal{H}(1-3w)v + \frac{w'}{1+w}v + \frac{\delta P}{\bar\rho(1+w)} + \Phi = 0$$

* Poisson equation

$$\nabla^2\Phi = -4\pi G a^2\bar\rho\Big[\delta - 3\mathcal{H}(1+w)\nabla^2 v\Big]$$

define $\Delta = \delta - 3\mathcal{H}(1+w)\nabla^2 v$

so

$$\nabla^2\Phi = 4\pi G\bar\rho a^2\Delta$$

$$\nabla^2\Phi = 4\pi G\bar{\rho}a^2\Delta$$

* Dark matter perturbations, pressureless, using Δ :

$$\Delta' + \nabla^2 v = 0$$

$$v' + \mathcal{H}v + \Phi = 0$$

$$\nabla^2\Phi = 4\pi G\bar{\rho}a^2\Delta$$

* Dark matter perturbations, pressureless, using Δ:

$$\Delta' + \nabla^2 v = 0$$

$$v' + \mathcal{H}v + \Phi = 0$$

* In this limit, agrees with non-relativistic perturbation theory

$$\dot{\delta} + \vec{\nabla}\cdot\vec{v} = 0$$

$$\dot{\vec{v}} + H\vec{v} = -\vec{\nabla}\Phi$$

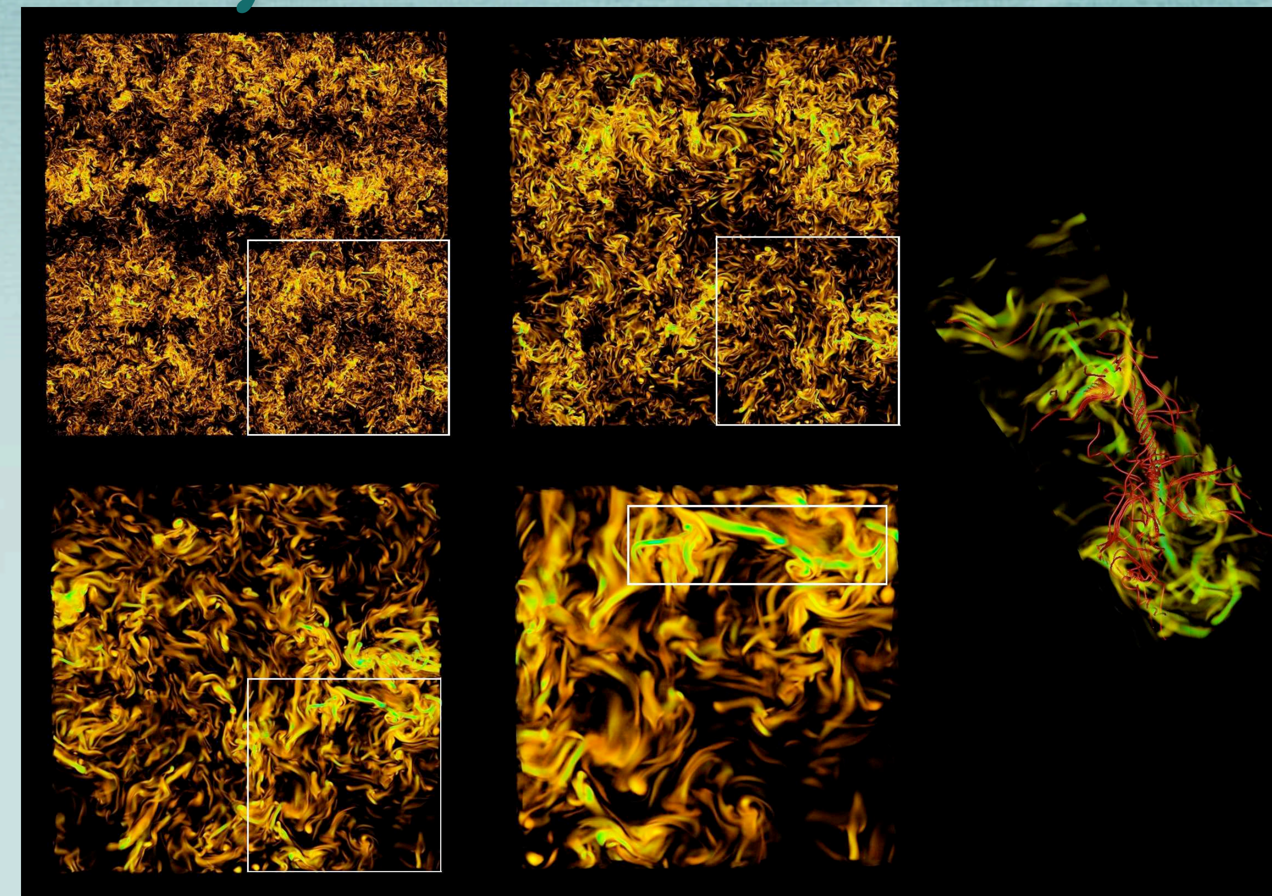
$$\nabla^2\Phi = 4\pi G\bar{\rho}a^2\delta$$

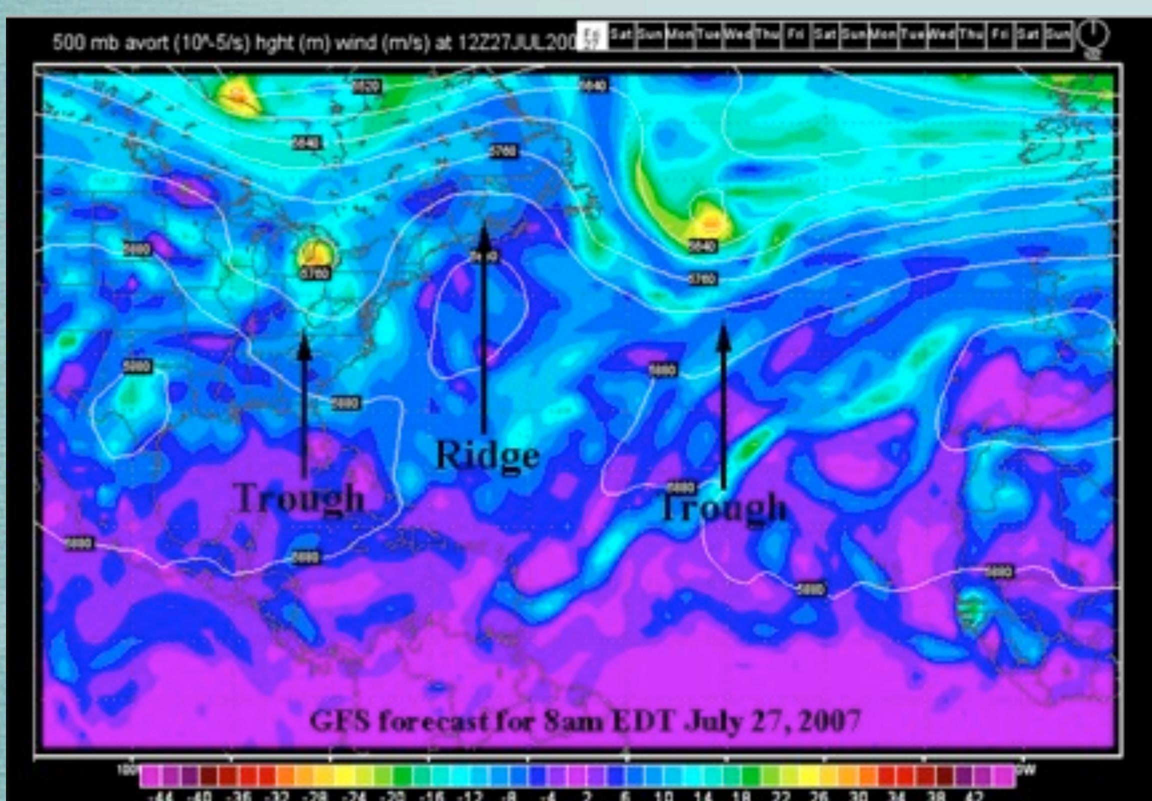
When Newtonian theory is not enough...

* But Newtonian theory cannot model
  - perturbations in relativistic species (radiation, neutrinos,...)
  - regions of high pressure (eg early universe)
  - regions of a comparable size of the horizon

* Effects of relativity on initial condition generation for N-body simulations?
  - Work in progress with Hidalgo ++

Vorticity





Vorticity in fluid dynamics

- Classical fluid dynamics $\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}$
- Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla P$$

- Evolution:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}) + \frac{1}{\rho^2}\nabla\rho \times \nabla P$$

  - 'source' term zero if ∇P and $\nabla\rho$ are parallel
  - i.e. barotropic fluid, no source term
- The inclusion of **entropy** provides a source for vorticity

Crocco (1937)

Entropy perturbations

* Adiabatic system $$\frac{\delta P}{\dot{P}} = \frac{\delta \rho}{\dot{\rho}}$$

* Non-adiabatic system allows for entropy perturbations

$$\frac{\delta P}{\dot{P}} \neq \frac{\delta \rho}{\dot{\rho}} \quad \Longrightarrow \quad \delta P = \frac{\dot{P}}{\dot{\rho}} \delta \rho + \delta P_{\mathrm{nad}}$$

* These entropy perturbations naturally occur in any system containing more than one component, such as
  - Standard cosmological fluid (relativistic vs. non-rel matter)
  - Models of cosmological inflation

Linear vorticity in cosmology

* First order vorticity evolves as

$$\omega'_{1ij} - 3\mathcal{H}c_{\rm s}^2\omega_{1ij} = 0$$

Kodama & Sasaki (1984)

* Reproduces well known result that, in radiation domination,

$$|\omega_{1ij}{\omega_1}^{ij}| \propto a^{-2}$$

* i.e. in absence of anisotropic stress, no source term: $\omega_{1ij} = 0$ is a solution to the evolution equation

Beyond linear perturbation theory

* Can go beyond the linear approximation by expanding small perturbations in a series, e.g.,

$$\delta\rho(\vec{x},t) = \bar{\rho}(t) + \delta\rho_1(\vec{x},t) + \frac{1}{2}\delta\rho_2(\vec{x},t)$$

  where $\delta\rho_2 < \delta\rho_1 < \bar{\rho}$

* In linear perturbation theory scalars decouple from vectors and tensors

* Crucial difference at higher orders: vectors, e.g., can be sourced by couplings between scalars.

Vorticity evolution: second order

* Second order vorticity, ω_{2ij}, evolves as

$$\omega'_{2ij} - 3\mathcal{H}c_{\rm s}^2\omega_{2ij} = \frac{2a}{\rho_0 + P_0}\left\{3\mathcal{H}V_{1[i}\delta P_{{\rm nad}1,j]} + \frac{\delta\rho_{1,[j}\delta P_{{\rm nad}1,i]}}{\rho_0 + P_0}\right\}$$

assuming zero first order vorticity.

* Including entropy gives a non-zero source term

AJC, Malik & Matravers (2009)

cf.

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}) + \frac{1}{\rho^2}\nabla\rho \times \nabla P$$

* This generalises Crocco's theorem to an expanding framework

'Estimating' the power spectrum

* Work in radiation era, and define the power spectrum as

$$\langle \omega_2^*(\boldsymbol{k}_1, \eta)\omega_2(\boldsymbol{k}_2, \eta)\rangle = \frac{2\pi}{k^3}\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2)\mathcal{P}_\omega(k, \eta)$$

* For the inputs:
  - Can solve linear equation for $\delta\rho_1$; leading order for small $k\eta$

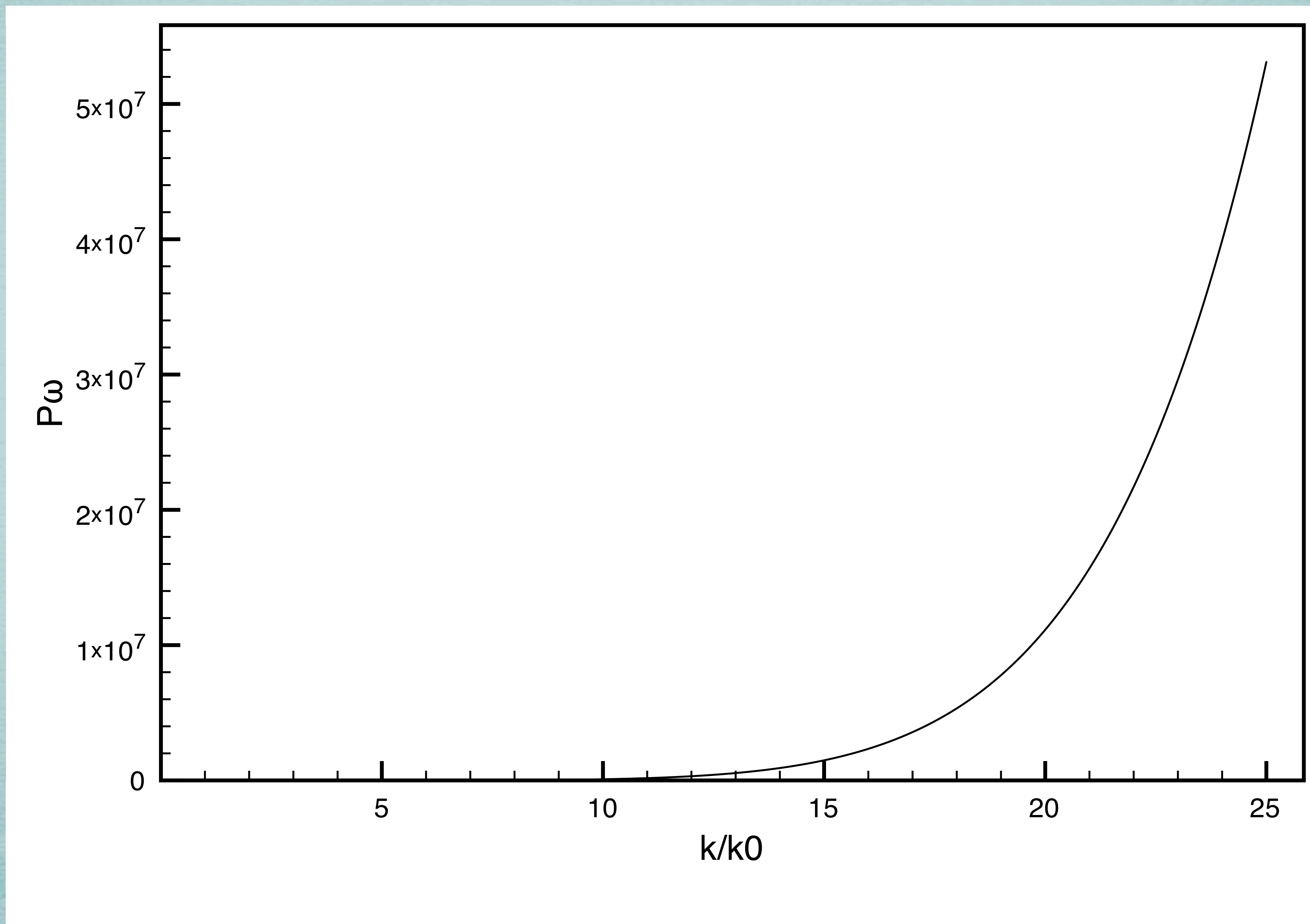
$$\delta\rho_1(k, \eta) = A\left(\frac{k}{k_0}\right)\left(\frac{\eta}{\eta_0}\right)^{-4}$$

  - 'Ansatz' for non-adiabatic pressure

$$\delta P_{\mathrm{nad}1}(k, \eta) = D\left(\frac{k}{k_0}\right)^2\left(\frac{\eta}{\eta_0}\right)^{-5}$$

* These give the spectrum

$$\frac{\mathcal{P}_\omega}{\mathrm{Mpc}^4} \sim 0.87 \times 10^{-2} \left(\frac{k}{k_0}\right)^7 + 3.73 \times 10^{-11} \left(\frac{k}{k_0}\right)^9 - 7.71 \times 10^{-20} \left(\frac{k}{k_0}\right)^{11}$$



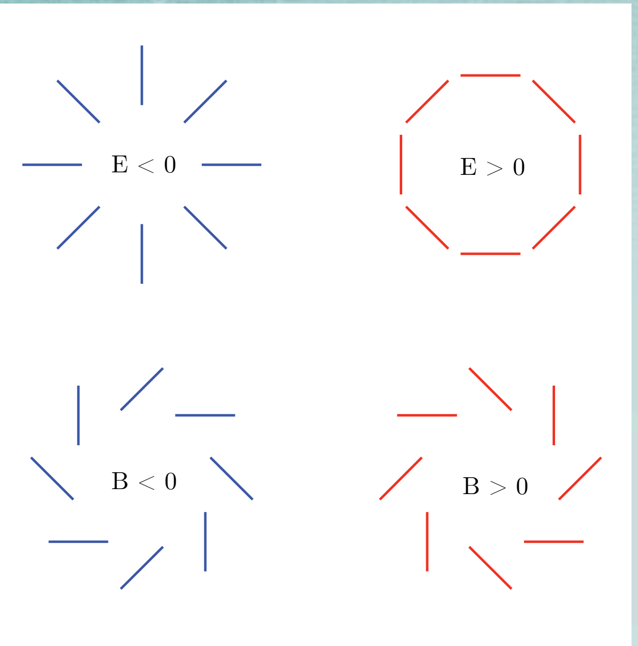
AJC, Malik & Matravers (2011)

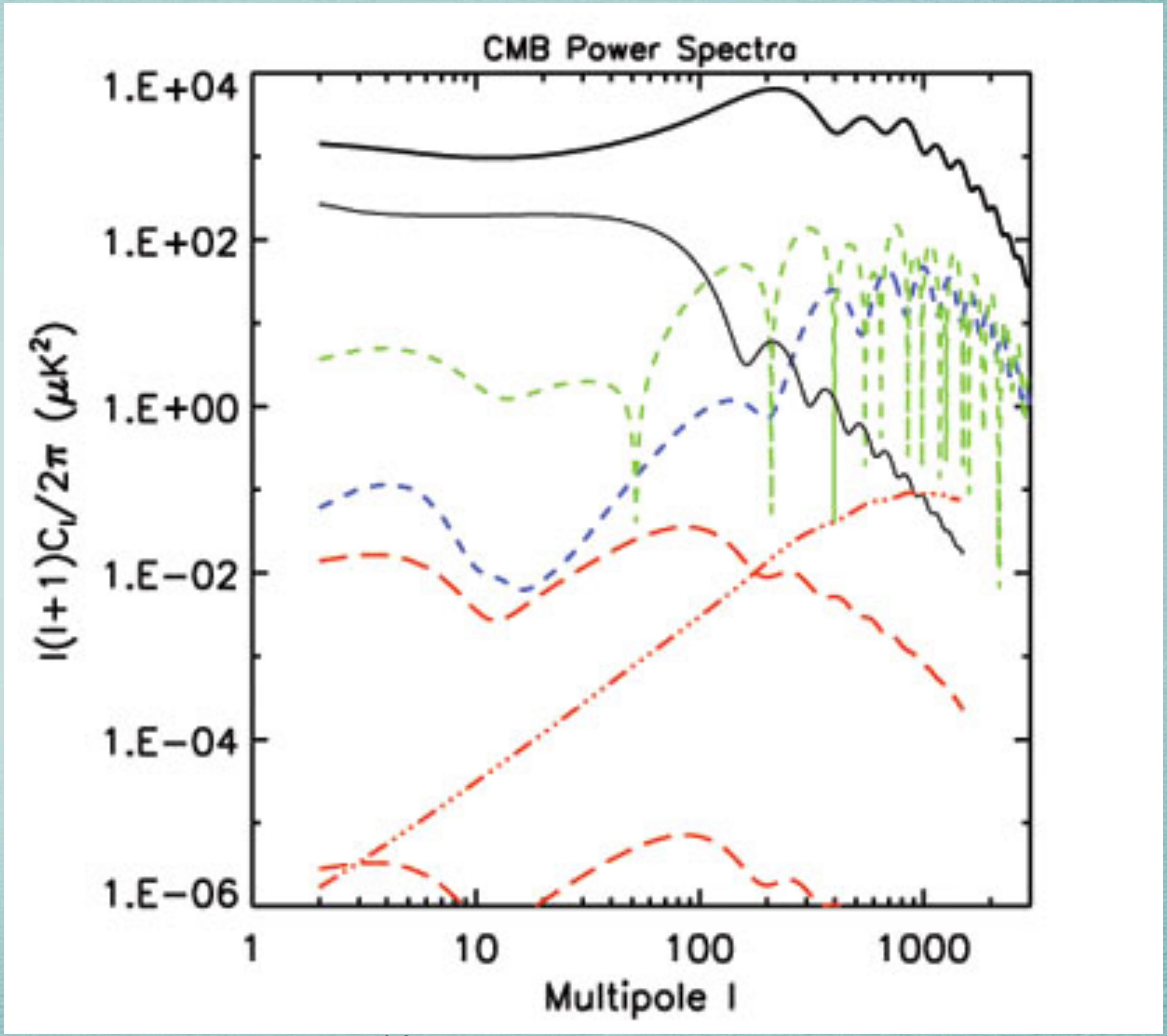
Observational signatures

* For linear perturbations, B mode polarisation of the CMB only produced by tensor perturbations:
  - scalar perturbations only produce E mode polarisation
  - vectors produce B modes, but decay with expansion
* Second order, vector perturbations produced by first order density and entropy perturbations source B mode polarisation
* Important for current and future CMB polarisation expts
* Could prove important for studying physics of primordial magnetic fields

Fenu et. al. (2011)

Observational signatures





Future directions

* Aim to go beyond rough approximation of power spectrum

* Require non-adiabatic pressure perturbation from realistic scenarios, e.g.

  - Relative entropy in concordance cosmology Brown, AJC & Malik (2011)

  - Isocurvature in multiple inflation models Huston & AJC (2011)

* Investigate potential of second order vorticity to source primordial magnetic seed fields.

Summary

- Universe is well described by a homogeneous and isotropic FLRW background + perturbations
- Inhomogeneous perturbations described using cosmological perturbation theory
- In non-relativistic regime, this corresponds to Newtonian perturbation theory (with a suitable choice of variables)
- Vorticity can be generated at second order in cosmological perturbations, sourced by entropy perturbations
- Could prove important for B-mode CMB polarisation, or for sourcing primordial magnetic fields.