THE ORIGIN OF COSMIC STRUCTURE

Carlos S. Frenk
Institute of Computational Cosmology, Physics Department, University of Durham, England

Abstract
A timely combination of new theoretical ideas and observational discoveries has brought about significant advances in our understanding of cosmic evolution. In the current paradigm, our Universe has a flat geometry, is undergoing accelerated expansion and is gravitationally dominated by elementary particles that make up cold dark matter. Within this framework, it is possible to calculate the emergence of galaxies and other structures from small quantum fluctuations imprinted during an epoch of inflationary expansion shortly after the Big Bang. I review the basic concepts that underlie our understanding of the formation of cosmic structure and some of the tools required to calculate it. Although many unresolved questions remain, a coherent picture for the formation of cosmic structure in now beginning to emerge. The notes associated with these lectures may be found at http://star-www.dur.ac.uk/∼csf/homepage/GalForm_lectures

1. INTRODUCTION

The origin of structure in the Universe is a central problem in Physics. Its solution will not only inform our understanding of the processes by which matter became organized into galaxies and clusters, but it will also help uncover the identity of the dark matter, offer insights into events that happened in the early stages of the Big Bang and provide a useful check on the values of the fundamental cosmological parameters estimated by other means.

The problem of the origin of cosmic structure is well posed because the initial conditions – small perturbations in the density and velocity field of matter – are, in principle, known from Big Bang theory and observations of the early Universe, while the basic physical principles involved are understood. The behaviour of the dark matter is governed primarily by gravity, while the formation of the visible parts of galaxies involves gas dynamics and radiative processes of various kinds. The early stages of evolution of the perturbations can be followed using linear theory, but the late stages require large computer simulations. In this way, it is possible to follow the development of structure from primordial pertubations to the point where the model can be compared with observations.

Over the past few years, there has been huge progress in quantifying observationally the properties of galaxies not only in the nearby universe, but also in the very distant universe. Since the clustering pattern of galaxies is rich with information about physics and cosmology, much effort is invested in mapping the distribution of galaxies at different epochs. Two large surveys, the US-based Sloan Digital Sky Survey [1] and the Anglo-Australian "2-degree field galaxy redshift survey" (2dFGRS) [2], are revolutionizing our view of the nearby universe with order of magnitude increases in the amount of available data. Similarly, new data collected in the past decade or so have opened up the high redshift universe[1] to detailed statistical study [3].

The advent of large computers, particularly parallel supercomputers, together with the development of efficient algorithms, has enabled the accuracy and realism of simulations to keep pace with observational progress. With the wealth of data now available, simulations are essential to interpret astronomical data and to link them to physical and cosmological theory.

To build a model of large-scale structure, four key ingredients need to be specified: (i) the content of Universe, (ii) the initial conditions, (iii) the growth mechanism, and (iv) the values of fundamental cosmological parameters. I now discuss each of these in turn.

1.1 The content of the Universe

Densities are usually expressed in terms of the cosmological density parameter, $\Omega = \rho/\rho_{crit}$, where the critical density, ρ_{crit}, is the value that makes the geometry of the Universe flat. The main constituents of the Universe and their contribution to Ω are listed in Table 1.

Table 1. *The content of the Universe*

Component	**Contribution to Ω**
CMB radiation	$\Omega_r = 4.7 \times 10^{-5}$
massless neutrinos	$\Omega_\nu = 3 \times 10^{-5}$
massive neutrinos	$\Omega_\nu = 6 \times 10^{-2} (\frac{<m_\nu>}{1\mathrm{ev}})$
baryons	$\Omega_b = 0.037 \pm 0.009$
(of which stars)	$\Omega_s = (0.0023 - 0.0023) \pm 0.0003$
dark matter	$\Omega_{\mathrm{dm}} \simeq 0.26 \pm 0.04$
dark energy	$\Omega_\Lambda \simeq 0.70 \pm 0.04$

The main contribution to the extragalactic radiation field today is the cosmic microwave background (CMB), the redshifted radiation left over from the Big Bang. These photons have been propagating freely since the epoch of "recombination", approximately 300,000 years after the Big Bang. The CMB provides a direct observational window to the conditions that prevailed in the early Universe. The Big Bang also produced neutrinos which today have an abundance comparable to that of photons. We do not yet know for certain what, if any, is the mass of the neutrino, but even for the largest masses that seem plausible at present, ~ 0.1eV, neutrinos make a negligible contribution to the total mass budget (although they could be as important as baryons). The abundance of baryons, dark matter and dark energy can be inferred, as discussed below, by combining data on inhomogeneities measured in the CMB with data on galaxy clustering [4]. Independent estimates of Ω_b can be obtained with reasonable precision by comparing the abundance of deuterium predicted by Big Bang theory with observations of the absorption lines produced by intergalactic gas clouds at high redshift seen along the line-of-sight to quasars [5]. Baryons, the overwhelming majority of which are *not* in stars today, are also dynamically unimportant (except, perhaps, in the cores of galaxies).

[1] In cosmology, distances to galaxies are estimated from the redshift of their spectral lines; higher redshifts correspond to more distant galaxies and thus to earlier epochs.

Dark matter makes up most of the matter content of the Universe today. To the now firm dynamical evidence for its existence in galaxy halos, even more direct evidence has been added by the phenomenon of gravitational lensing which has now been detected around galaxy halos (e.g. [6, 7]), in galaxy clusters (e.g. [8]), and in the general mass field (e.g. [9] and references therein). The distribution of dark matter in rich clusters can be reconstructed in fair detail from the weak lensing of distant background galaxies in what amounts virtually to imaging the cluster dark matter. Various dynamical tests give values of $\Omega_{\rm dm} \simeq 0.3$, consistent with the CMB estimates and also with other, independent determinations such as those based on the baryon fraction in clusters ([10, 11]), and on the evolution in the abundance of galaxy clusters ([12, 13]). Since $\Omega_{\rm dm}$ is much larger than Ω_b, it follows that the dark matter cannot be made of baryons. The most popular candidate for the dark matter is a hypothetical elementary particle like those predicted by supersymmetric theories of particle physics. These particles are referred to generically as cold dark matter or CDM. (Hot dark matter is also possible, for example, if the neutrino had a mass of ~ 5 eV. However, early cosmological simulations showed that the galaxy distribution in a universe dominated by hot dark matter would not resemble that observed in our Universe [14].

A recent addition to the cosmic budget is the dark energy, evidence for which was first provided by studies of type Ia supernovae [15,16] [2]. These presumed 'standard candles' can now be observed at redshifts between 0.5 and 1 and beyond. Those at z $(0.3 - 1)$ are fainter than would be expected if the universal expansion were decelerating today, indicating that the expansion is, in fact, accelerating. Within the standard Friedmann cosmology, there is only one agent that can produce an accelerating expansion. This is nowadays known as dark energy, a generalization of the cosmological constant first introduced by Einstein, which could, in principle, vary with time. The supernova evidence is consistent with the value $\Omega_\Lambda \simeq 0.7$. Further, independent evidence for dark energy is provided by a recent joint analysis of CMB data (see next section) and the 2dFGRS [30].

Amazingly, when all the components are added together, the data are consistent with a flat universe, $\Omega_{\rm tot} = 1$.

1.2 The initial conditions

The idea that galaxies and other cosmic structures are the result of the slow amplification by the force of gravity of small primordial perturbations present in the mass density at early times goes back, at least, to the 1940s [19]. However, it was only in the early 1980s that a physical mechanism capable of producing small perturbations was identified. This is the mechanism of inflation, an idea due to Guth [20], which changed the face of modern cosmology. Inflation is produced by the dominant presence of a quantum scalar field which rolls slowly from a false to the true vacuum, maintaining its energy density approximately constant and causing the early Universe to expand exponentially for a brief period of time. Quantum fluctuations in the inflaton field are blown up to macroscopic scales and become established as genuine adiabatic ripples in the energy density. Simple models of inflation predict the general properties of the resulting fluctuation field: it has Gaussian distributed amplitudes and a near scale-invariant power spectrum [21].

[2] The possibility that dark energy might be the dynamically dominant component had been anticipated by theorists from studies of the cosmic large-scale structure (see e.g. [17]), and was considered in the first simulations of structure formation in cold dark matter universes [18].

After three decades of ever more sensitive searches, evidence for the presence of small fluctuations in the early universe was finally obtained in 1992. Since prior to recombination the matter and radiation fields were coupled, fluctuations in the mass density are reflected in the temperature of the radiation. Temperature fluctuations in the CMB were discovered by the COBE satellite [22] and are now being measured with ever increasing accuracy, particularly by detectors deployed in long-flight balloons [23, 24, 25]. The spectrum of temperature fluctuations is just what inflation predicts: it is scale invariant on large scales and shows a series of "Doppler" or "acoustic" peaks which are the result of coherent acoustic oscillations experienced by the photon-baryon fluid before recombination. The characteristics of these peaks depend on the values of the cosmological parameters. For example, the location of the first peak is primarily determined by the large-scale geometry of the Universe and thus by the value of Ω. Current data imply a flat geometry.

The spectrum of primordial fluctuations generated, for example, by inflation evolves with time in a manner that depends on the content of the Universe and the values of the cosmological parameters. The dark matter acts as a sort of filter, inhibiting the growth of certain wavelengths and promoting the growth of others. Following the classical work of Bardeen et al. [26], transfer functions for different kinds of dark matter (and different types of primordial fluctuation fields, including non-Gaussian cases) have been computed. In Gaussian models, the product (in Fourier space) of the primordial spectrum and the transfer function, together with the growing mode of the associated velocity field, provides the initial conditions for the formation of cosmic structure.

1.3 Growth mechanism

Primordial fluctuations grow by gravitational instability: overdense fluctuations expand linearly, at a retarded rate relative to the Universe as a whole, until eventually they reach a maximum size and collapse non-linearly to form an equilibrium (or 'virialized') object whose radius is approximately half the physical size of the perturbation at maximum expansion. The theory of fluctuation growth is lucidly explained by Peebles [27].

Although gravitational instability is now widely accepted as the primary growth mechanism responsible for the formation of structure, it is only very recently that firm empirical evidence for this process was found. Gravitational instability causes inflow of material around overdense regions. From the perspective of a distant observer, this flow gives rise to a characteristic infall pattern which is, in principle, measurable in a galaxy redshift survey by comparing the two-point galaxy correlation function along and perpendicular to the line-of-sight. In this space, the infall pattern resembles a butterfly [28]. This pattern has been clearly seen for the first time in the 2dFGRS [29][3].

1.4 Cosmological parameters

After decades of debate, the values of the fundamental cosmological parameters are finally being measured with some degree of precision. The main reason for this is the accurate measurement of the acoustic peaks in the CMB temperature anisotropy

[3] Strictly speaking the 'butterfly' pattern does not prove the existence of infall since the continuity equation would ensure a similar pattern even if velocities were induced by non-gravitational processes. However, it can be shown that such velocities, if present, would rapidly decay.

spectrum whose location, height and shape depend on the values of the cosmological parameters. Some parameter degeneracies exist but some of these can be broken using other data, for example, the distant Type Ia supernovae or the 2dFGRS (eg. [30]). The CMB data alone do not constrain the Hubble constant, but these data combined with the 2dFGRS give a value, in units of 100 km s^{-1} Mpc,$^{-1}$, of $h = 0.68 \pm 0.07$, in agreement with results from the HST key project [31], and other methods. In addition to h and the other parameters listed in Table 1, the other important number in studies of large-scale structure is the amplitude of primordial density fluctuations which is usually parametrized by the quantity σ_8 (the linearly extrapolated value of the top-hat filtered fluctuation amplitude on the fiducial scale of 8 $h^{-1}\,\mathrm{Mpc}$). The best estimate of this quantity comes from the observed abundance of rich galaxy clusters which gives $\sigma_8\Omega^{0.6} = 0.5$, with an uncertainty of about 10% [32, 33, 34].

2. THE FRIEDMANN MODEL

This section provides is a very brief review of the main equations that describe the evolution of the mean properties of the universe.

The separation of two points can be written as

$$\overline{r} = a(t)\overline{x} \tag{2.1}$$

where $\overline{x}$ is the *comoving separation* and is constant for two points moving with the mean expansion; a is the scale factor ($a = 1$ today) and is related to redshift z by $a = 1/(1+z)$.

On large scales, the Universe is expanding. The relative velocity of two comoving objects, $\overline{u}$, is proportional to their physical separation, $\overline{r}$, by:

$$\overline{u} = H(t)\overline{r} \tag{2.2}$$

The Hubble parameter, H, is uniform in space and is expressed as:

$$H_0 = 100h\mathrm{km/s} \tag{2.3}$$

where h is measured to be $h = 0.73 \pm 0.06$.

Differentiating (2.1) wrt time:

$$\overline{u} \equiv \dot{\overline{r}} = \dot{a}\overline{x} = \frac{\dot{a}}{a}a\overline{x} = \frac{\dot{a}}{a}\overline{r} \tag{2.4}$$

which, from (2.2), implies

$$H = \frac{\dot{a}}{a} \tag{2.5}$$

The evolution of H is described by Einstein's equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi}{3}\mathrm{G}\overline{\rho} + \frac{\Lambda c^2}{3} \tag{2.6}$$

and

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi\mathrm{G}}{3}\left(\overline{\rho} + \frac{3\overline{P}}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{2.7}$$

where a is the expansion factor of the universe, k the curvature, ρ the mass-energy density, Λ the cosmological constant and $H = \dot{a}/a$ is the Hubble constant. These dynamical equations relate the velocity and acceleration of the Universe to its matter content of density ρ and pressure p. The equation of state is:

$$p = w\rho c^2 \tag{2.8}$$

For radiation, $w = 1/3$. The cosmological constant is often regarded as a scalar field with $w = -1$. Current observations require a non-zero value of Λ.

The 1^{st} law of thermodynamics, $dU + pdV = 0$, implies that:

$$\frac{d\rho}{dt} + 3H\left(\rho + \frac{P}{c^2}\right) = 0 \tag{2.9}$$

which can be easily integrated to show that for non-relativistic (cold) matter, $\rho_m \propto a^{-3}$, whereas for relativistic particles or radiation, $\rho_r \propto a^{-4}$.

Setting $\rho = \rho_\gamma + \rho_m$ in (2.6),

$$\Omega_m + \Omega_\gamma + \Omega_\Lambda = 1 + k\left(\frac{c}{aH}\right)^2 \tag{2.10}$$

where the cosmological density parameter, $\Omega = \rho/\rho_{crit}$, where $\rho_{crit} = 3H^2/8\pi G$ and

$$\Omega_m = \frac{8\pi}{3H^2}G\rho_m, \quad \Omega_\gamma = \frac{8\pi}{3H^2}G\rho_\gamma, \quad \Omega_\Lambda = \frac{\Lambda c^2}{3H^2} \tag{2.11}$$

denote the contributions from matter, radiation and cosmological constant. The constant k describes the curvature of the Universe and can be 0 or ± 1. Observations of fluctuations in the cosmic microwave background suggest $k = 0$. (For most of this course, we will consider only the $k = 0$ model.) In this case.

$$\Omega_m + \Omega_\gamma + \Omega_\Lambda = 1 \tag{2.12}$$

The relative importance of the three at different epochs may be seen by writing $\frac{\Omega_\Lambda}{\Omega_m} = \frac{\Omega_{\Lambda 0}}{\Omega_{m0}}a^3$, etc and then writing one of the Ωs eg Ω_m as:

$$\Omega_m = \frac{\Omega_{m0}}{\Omega_{m0} + a^{-1}\Omega_{\gamma_0} + a^3\Omega_{\Lambda_0}} \tag{2.13}$$

From this, we see that the Universe goes through three stages of evolution:

- Radiation-dominated for $a \lesssim 1.4 \times 10^{-4}$
- Matter-dominated for $1.4 \times 10^{-4} \lesssim a \lesssim 0.8$
- Λ-dominated for $a \gtrsim 0.8$

It is a mystery why we live so close to the transition from matter to Λ-domination

3. THE GROWTH OF DENSITY FLUCTUATIONS – LINEAR THEORY

3.1 Linear Fluctuations

To study the evolution of linear perturbations, we write the density, pressure and velocity fields, $\rho(\overline{r},t)$, $P(\overline{r},t)$ and $\overline{u}(\overline{r},t)$, as a mean value plus an initially small fluctuation:

$$\rho(\overline{r},t) = \overline{\rho}(t) + \delta\rho(\overline{r},t) = \overline{\rho}(t)(1+\delta(\overline{r},t)), \tag{3.1}$$

where $\delta = \delta\rho/\overline{\rho}$

$$P(\overline{r},t) = \overline{P}(t) + \delta P(\overline{r},t)$$
$$\overline{u} = H(t)\overline{r} + \overline{v}(\overline{r},t) \tag{3.2}$$

where **v** is the peculiar velocity. For linear fluctuations, $\delta \ll 1$ and $v \ll Hr$.

3.2 The fluctuation growth equation in the non-relativistic regime

For an ideal, non-relativistic fluid, the equations of conservation of mass and momentum are:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\overline{u}) = 0 \tag{3.3}$$

$$\frac{\partial\overline{u}}{\partial t} + \overline{u}\cdot\ \nabla\overline{u} = -\frac{\nabla P}{\rho} - \nabla\Phi \tag{3.4}$$

where Φ is the gravitational potential which, for a self-gravitating fluid, is given by Poisson's eqn:

$$\nabla^2\Phi = 4\pi G(\rho_{tot} + \frac{3P_{tot}}{c^2}) - \Lambda c^2 \tag{3.5}$$

where the suffix *tot* indicates that all matter contributes to the gravitational potential whereas the conservation equations apply separately to the baryonic fluid and also to any component of non-baryonic dark matter. (Note that for radiation, $3P_{tot}/c^2 = \rho_\gamma$ so in the last equation, the P term is related to ρ_γ). (For a derivation of the fluid equations, see eg Binney & Tremaine *Galactic Dynamics*).

Substituting (3.1) and (3.2) into (3.3) gives

$$\dot{\overline{\rho}}(1+\delta) + \overline{\rho}\dot{\delta} + \overline{\rho}\nabla\cdot((1+\delta)(H\overline{r}+\overline{v})) = 0 \tag{3.6}$$

Using $\nabla\cdot\overline{r} = 3$, the non-perturbative terms in this equation reduce to (2.9) with $P/c^2 = 0$ (which is appropriate for a non-relativistic fluid). Thus, the linear equation relating the terms of first order in δ and v/Hr is:

$$\dot{\delta} + H\overline{r}\cdot\nabla\delta + \nabla\cdot\overline{v} = 0. \tag{3.7}$$

Similarly, from (3.4), using the fact that $\nabla\overline{r}$ is the identity matrix,

$$\dot{H}\overline{r} + \frac{\partial\overline{v}}{\partial t} + (H\overline{r}+\overline{v})\cdot(H+\nabla\overline{v}) = -\frac{\nabla(\overline{P}+\delta P)}{\overline{\rho}(1+\delta)} - \nabla(\overline{\Phi}+\phi) \tag{3.8}$$

where $\phi \equiv \delta\phi$ is the perturbative part of the potential, $\Phi = \overline{\Phi} + \phi$. The non-perturbative part of this equation is:

$$\dot{H}\overline{r} + H^2\overline{r} = -\frac{\nabla\overline{P}}{\overline{\rho}} - \nabla\Phi \tag{3.9}$$

which, upon taking the divergence and using (3.5), gives:

$$3\dot{H}+3H^2=-4\pi G(\overline{\rho}+\frac{3\overline{P}}{c^2})+\Lambda c^2 \tag{3.10}$$

which is the same as (2.7). The first order terms give:

$$\frac{\partial\overline{v}}{\partial t}+H\overline{v}+H\overline{r}\cdot\nabla\overline{v}=-\frac{\nabla\delta P}{\overline{\rho}}-\nabla\phi \tag{3.11}$$

where ϕ is the perturbative part of Φ:

$$\nabla^2\phi=4\pi G(\delta\rho_{\rm tot}+\frac{3\delta P_{\rm tot}}{c^2}) \tag{3.12}$$

Eqns (3.7) and (3.11) can be simplified by using comoving coordinates, ie by changing from $(\overline{r},t)\to(\overline{x},t')$, where $\overline{r}=a\overline{x}$ and $t'=t$. We have:

$$\frac{\partial}{\partial t}\Big|_{\overline{r}}=\frac{\partial}{\partial t'}\Big|_{\overline{x}}\frac{\partial t'}{\partial t}\Big|_{\overline{r}}+\frac{\partial}{\partial\overline{x}}\Big|_{t}\frac{\partial\overline{x}}{\partial t}\Big|_{\overline{r}} \tag{3.13}$$

Now,

$$\frac{\partial\overline{x}}{\partial t}\Big|_{\overline{r}}=-\frac{\overline{x}a\dot{a}}{a^2} \tag{3.14}$$

so

$$\frac{\partial}{\partial t}\Big|_{\overline{r}}=\frac{\partial}{\partial t'}\Big|_{\overline{x}}-H\overline{x}\frac{\partial}{\partial\overline{x}}\Big|_{t'} \tag{3.15}$$

Similarly

$$\frac{\partial}{\partial\overline{x}}\Big|_{t}=\frac{\partial}{\partial\overline{r}}\Big|_{t'}\frac{\partial\overline{r}}{\partial\overline{x}}=a\frac{\partial}{\partial\overline{r}}\Big|_{t'} \tag{3.16}$$

ie

$$\nabla_{\overline{x}}=a\nabla_{\overline{r}} \tag{3.17}$$

Eqns (3.7) and (3.11) then become:

$$\frac{\partial\delta}{\partial t}\Big|_{\overline{r}}+H\overline{r}\cdot\nabla\delta+\nabla\cdot\overline{v}=\frac{\partial\delta}{\partial t}\Big|_{\overline{x}}-H\overline{x}\frac{\partial\delta}{\partial\overline{x}}\Big|_{t}+H\overline{r}\cdot\nabla\delta+\nabla\cdot\overline{v}=0$$

$$\to\frac{\partial\delta}{\partial t}\Big|_{\overline{x}}+\nabla_{\overline{r}}\cdot\overline{v}=0 \tag{3.18}$$

and

$$\frac{\partial\overline{v}}{\partial t}\Big|_{\overline{x}}+H\overline{v}=-\frac{\nabla_{\overline{r}}\delta P}{\overline{\rho}}-\nabla_{\overline{r}}\phi \tag{3.19}$$

To obtain our final equation, take the divergence of (3.19) and the time derivative of (3.18) and combine the two. This gives:

$$\frac{\partial^2\delta}{\partial t^2}\Big|_{\overline{x}}+2H\frac{\partial\delta}{\partial t}\Big|_{\overline{x}}=\frac{\nabla^2_{\overline{r}}\delta P}{\overline{\rho}}+\nabla^2_{\overline{r}}\phi \tag{3.20}$$

where

$$\nabla^2_{\overline{r}}\phi=4\pi G(\delta\rho_{\rm tot}+\frac{3\delta P_{\rm tot}}{c^2}) \tag{3.21}$$

Note:

• Eqn (3.20) applies to any component of non-relativistic (cold) matter ($P\ll\frac{1}{3}\rho c^2$); δ refers to fluctuations in density and δP in pressure

• $\delta\rho_{\rm tot}$ and $\delta P_{\rm tot}$, on the other hand, are the sources of perturbation in the gravitational potential and so refer to the sum of all components of matter.

• The $2H\frac{\partial\delta}{\partial t}$ term is a cosmological drag term. It is the only term where the expansion of the universe comes in. Thus, the Hubble expansion tends to dampen fluctuations. Without this term, fluctuations would grow exponentially; with it, growth is a power-law at best.

3.3 Solution for cold matter (dust)

We will now solve (3.20) for the case where pressure fluctuations are zero. This is known as the "dust solution" and corresponds to cold matter. Write:

$$\rho_{\rm tot} = \rho_m + \rho_r \tag{3.22}$$

and set $\delta P_{\rm tot} = 0$. Then, $\delta\rho_{\rm tot} = \delta\rho_m \equiv \rho_m\delta$. (Note that $\delta\rho_r = 0$ because we are in the non-relativistic regime, so ρ_r contributes to $\rho_{\rm tot}$ but is not perturbed.) Eqn (3.20) becomes:

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\rho_m\delta \tag{3.23}$$

where the dot denotes the time derivative at constant $\overline{x}$. Using (1.12), we can write this as

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\Omega_m\delta = 0 \tag{3.24}$$

Let us consider the solution in 3 different situations:

• Matter-dominated case, $\Omega \simeq \Omega_m \simeq 1$. This is the most important case since it describes the Universe since recombination until almost the present day. Setting $\Omega = 1$ in (2.11), we have $H^2 \propto \rho_m \propto a^{-3}$, from which it is easy to show that $H = \frac{2}{3t}$. Substituting in (3.24),

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0 \tag{3.25}$$

The general solution is:

$$\delta = A(\overline{x})t^{2/3} + B(\overline{x})t^{-1} \tag{3.26}$$

where A and B are constants. The first term is known as the "growing" mode and the second as the "decaying" mode. The latter may be neglected since any decaying modes generated in the early universe will become negligible by the time of recombination. It is usual to normalize the growing mode to the value it would have today if the fluctuation had been growing at the linear rate:

$$\delta = \delta_0(\overline{x})\left(\frac{t}{t_0}\right)^{2/3} = \delta_0(\overline{x})a \tag{3.27}$$

• Λ-dominated; $\Omega_\Lambda \simeq 1\,(\Omega_m \simeq 0)$. Although the Universe has not yet reached this state, it is rapidly approaching it. Eqn (2.11) gives $H^2 = \frac{\Lambda c^2\pi G}{3} = \text{const}$, so (3.24) gives

$$\ddot{\delta} + 2H\dot{\delta} = 0 \tag{3.28}$$

whose solution is

$$\delta = A(\overline{x}) + B(\overline{x})e^{-2Ht} \tag{3.29}$$

Thus, the growing mode is no longer growing, but has frozen in at a constant value.

• General case; $\Omega_m + \Omega_\Lambda = 1$. We will not attempt to solve this here, but a feel for the solution may be obtained by combining the two previous results. We write:

$$\delta = \delta_0(\overline{x})ag(a, \Omega_{m0}) \tag{3.30}$$

where $g(a, \Omega_{m0})$ is a correction factor that modifies the simple scaling of the $\Omega = 1$ case. The factor g is constant at early times ($\Omega \to 1$) and scales as $1/a$ at late times ($\Omega_\Lambda \to 1$). For $\Omega_m = 0.33$, one can show that $g \to g_i = 1.25$ as $a \to 0$. Thus, fluctuations in the matter distribution have grown by less than a factor of $\frac{1}{1.25a_{rec}} \simeq 800$ since recombination. Since the cosmic microwave background radiation indicates that fluctuations in the *baryon* distribution at recombination had amplitude of less than 10^{-4}, the existence of non-linear structures today ($\delta > 1$) implies that the growth of fluctuations must have been driven by non-baryonic dark matter which was not relativistic at the epoch of recombination, t_{rec}, and does not couple to the radiation. After t_{rec}, the baryons decouple and are free to fall into the dark matter potential wells.

3.4 The Meszaros effect

A perturbation in a *collisionless*, *non-relativistic* matter component (eg cold dark matter) experiences reduced growth during the period when the Universe is dominated by a relativistic component, eg radiation. To see this, let us consider eqn (3.20) for a non-relativistic matter component, which we will take to have $P = \delta P = 0$ (i.e. cold dark matter). The total mean density is $\overline{\rho}_{tot} = \overline{\rho}_{nr} + \overline{\rho}_r$ (with $\overline{\rho}_r > \overline{\rho}_{nr}$). The perturbed (non-relativistic) density is $\delta_{nr} = \delta\overline{\rho}_{nr}$. The relativistic (radiation) field is assumed to be uniform, $\delta_r = 0$, so $\delta\rho_{tot} = \delta_{nr} = \delta\overline{\rho}_{nr}$. Eqn.(3.20) then becomes (droping the overbars in $\overline{\rho}$, etc for clarity):

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_{nr}\delta = 0 \tag{3.31}$$

This is just like eqn.(3.23), but now we are interested in finding a solution which applies in both the radiation-dominated and matter-dominated regimes, and

Fig. 1: The correction factor g and linear growth factor δ as functions of expansion factor a, for density perturbations in a spatially flat universe dominated by cold matter and a cosmological constant.

in the transition between them. This means that both H and ρ_{nr} have more complicated dependences on a than for the matter-dominated regime assumed in eqn.(3.25).

Changing variables to

$$y = \frac{\rho_{nr}}{\rho_r} = \frac{a}{a_{eq}} \tag{3.32}$$

where a_{eq} is the expansion factor at the epoch of matter - radiation equality. Using the Friedmann equation (2.6) with $k = 0$, $\Lambda = 0$ and $\rho = \rho_{nr} + \rho_r$ and setting $\frac{d}{dt} = \dot{a}\frac{d}{da}$, eqn (3.31) becomes:

$$\frac{d^2\delta}{d^2y} + \frac{2+3y}{2y(1+y)}\frac{d\delta}{dy} - \frac{3y}{2y(1+y)} = 0. \tag{3.33}$$

which, as usual, has a growing and a decaying solution. The growing solution is

$$\delta_+ \propto 1 + \frac{3}{2}y \tag{3.34}$$

Before z_{eq} ($y < 1$), the growing mode is practically frozen. The total growth in the interval $t = 0 - t_{eq}$ is

$$\frac{\delta_+(y=1)}{\delta_+(y=0)} = \frac{5}{2} \tag{3.35}$$

After z_{eq}, the solution rapidly matches the growth rate in a matter-dominated Einstein-de-Sitter model,

$$\delta_+(y >> 1) \propto y \propto a \propto t^{2/3} \tag{3.36}$$

Physically, the explanation for the Meszaros effect is this. At early times, the dominant energy in radiation drives the Universe to expand so fast that the matter has no time to respond and δ is frozen.

As the radiation becomes negligible, growth increases smoothly to the behaviour appropriate in an Einstein-de-Sitter universe.

3.5 Fluctuations in a baryon universe

Consider fluctuations in the baryon distribution which we will take to be a perfect fluid. Assume that the fluctuations are adiabatic, ie that the cooling time is long compared with the age of the Universe. Then, $\delta P = c_s^2 \delta\rho$, where c_s is the sound speed. The pressure is greater in overdense regions and this impedes the growth of the fluctuation. The Jeans mass is derived by comparing the relative strength of gravitational and pressure forces acting on a perturbation. If, for an adiabatic perturbation, we write:

$$\delta P \propto \delta\rho \propto e^{i\overline{k}\cdot\overline{x}} \tag{3.37}$$

then,

$$\nabla_r^2 \delta P = \frac{1}{a^2}\nabla_x^2 \delta P = \frac{c_s^2}{a^2}\nabla_x^2 \delta\rho = -\frac{k^2 c_s^2}{a^2}\delta\rho \tag{3.38}$$

Substitute in (3.20), setting $\delta\rho = \overline{\rho}\delta$:

$$\ddot{\delta} + 2H\dot{\delta} = -\frac{k^2 c_s^2}{a^2}\delta + 4\pi G\left[\overline{\rho}\delta + 3\left(\frac{c_s}{c}\right)^2\overline{\rho}\delta\right]$$

We now suppose that the redshift is low enough that baryon density exceeds the radiation density ($\rho_b \gg \rho_r$). This implies that $c_s \ll c$, so that we can ignore the last term on the RHS, leading to

$$\ddot{\delta} + 2H\dot{\delta} = \left(4\pi \mathrm{G}\overline{\rho} - \frac{k^2 c_s^2}{a^2}\right)\delta \tag{3.39}$$

Fluctuations can only grow if RHS is positive. This can be seen, for example, by trying a solution of the form $\delta \propto e^{iwt}$. The resulting algebraic equation is called the dispersion relation. For fluctuations to grow, w must be imaginary and this requires the RHS of the equation to be positive. Thus fluctuations grow only if their physical wavelength, $\lambda = \frac{2\pi a}{k}$, is greater than the Jeans wavelength:

$$\lambda_J = c_s \left(\frac{\pi}{\mathrm{G}\overline{\rho}}\right)^{1/2} \tag{3.40}$$

corresponding to masses greater than the Jeans mass:

$$M_J = \frac{4\pi}{3}\overline{\rho}_b \left(\frac{\lambda_J}{2}\right)^3 = \frac{1}{6}\pi\overline{\rho}_b \left[c_s \left(\frac{\pi}{G\overline{\rho}}\right)^{1/2}\right]^3 \tag{3.41}$$

Note:

• This result is also valid for static clouds (Hubble drag term only slows down the rate of collapse) and can be used, for example, to determine the stability of molecular clouds in the Galaxy.

• The Jeans mass is defined for any mass component of density ρ_m (eg baryons, CDM, etc), but the mean density that comes into the Jeans length, $\overline{\rho}$, includes all gravitating components.

• At the epoch of matter-radiation equality, z_{eq} , $\rho_{nr} = \rho_{nr0}(1+z)^3$ is equal to $\rho_r = \rho_{r0}(1+z)^4$. This then gives $z_{eq} = 2.6 \times 10^4 \Omega_{m0} h^2$. At this epoch, the sounds speed is $\left(\frac{\partial P}{\partial \rho}\right)^{1/2} = \frac{c}{\sqrt{3}}$ and eqn (3.35) gives,

$$M_J = 3.5 \times 10^{15} (\Omega_{m0} h^2)^{-2}\, \mathrm{M}_\odot \tag{3.42}$$

• Using values appropriate for the period after recombination (and recalling that $\overline{\rho} \propto \rho \propto a^{-3}$ and $T \propto c_s^2 \propto a^{-1}$) gives:

$$M_J \simeq 5 a^{-3/2}\, \mathrm{M}_\odot \tag{3.43}$$

3.6.1 The horizon size and mass

So far, we have used Newtonian dynamics. In principle, fluctuations can have a wavelength larger than the horizon and, in this case, the Newtonian treatment breaks down. In fact, fluctuations generically go through a phase when they are larger than the horizon. We will not worry about the technical issue of whether the relevant horizon is the particle horizon ($R_{\rm horizon}$) or the Hubble radius (R_H), ie the radius of the speed-of-light sphere, where

$$R_H = \frac{c}{H}, \qquad (3.44)$$

$R_{\rm horizon}$ and R_H differ only by a factor of order unity. Physically, R_H is the typical size over which physical processes can operate coherently. Throughout most of the Universe's evolution, $a(t) \propto t^n$ with $n < 1$. Now, $H = \dot{a}/a \propto t^{-1}$ and $R_H \propto t$. Thus, R_H grows *faster* than proper length ($r \propto a \propto t^n$).

Example: Calculate the proper size today corresponding to the horizon at t_{eq}, the epoch at which the density in matter and radiation were equal. How does this compare with the Hubble radius today.

The epoch at which the density in matter and radiation are equal is:

$$1 + z_{eq} \simeq 2.6 \times 10^4 \Omega_{m0} h^2 \qquad (3.45)$$

corresponding to

$$t_{eq} \simeq 10^4 (\Omega_{m0} h^2)^{-2} \quad \text{yrs} \qquad (3.46)$$

where the contribution of 3 species of neutrinos has been included in the radiation density. Thus,

$$R_H(t_{eq}) = \frac{c}{H_0}\frac{t_{eq}}{t_0} \simeq 1.4 \times 10^{20} \quad \text{m} \quad (\simeq 5\text{kpc}) \tag{3.47}$$

A region of *proper* radius $R_H(t_{eq})$ at t_{eq} would today be

$$r_{eq}(z=0) = R_H(t_{eq})(1+z_{eq}) \simeq 100(\Omega_{m0}h^2)^{-1} \quad \text{Mpc} \tag{3.48}$$

which is much smaller than the Hubble radius today ($\simeq 3000\, h^{-1}$ Mpc).

Consider a perturbation of proper size λ, with $\lambda < R_H$ today. As we go back in time, the proper radius of this region shrinks as $a(t) \propto t^n$, with $n < 1$. But the Hubble radius of the universe decreases as t, ie *faster*. Thus, there will a time, $t = t_{enter}(\lambda)$, when the proper radius of the perturbation will equal R_H. For $t < t_{enter}$, this proper radius will be *larger* than the Hubble radius. We usually say that the lengthscale λ "enters the horizon" at $t_{enter}(\lambda)$

In analogy with the Jeans mass, define the horizon mass as:

$$M_H = \frac{1}{6}\pi\rho R_H^3 \tag{3.49}$$

where $\rho = \rho_m + \rho_r$. The baryonic part of this is:

$$M_{Hb} = \frac{1}{6}\pi\rho_b R_H^3 \tag{3.50}$$

Recalling that, for a flat, radiation-dominated universe,

$$H = \frac{1}{2t} \tag{3.51}$$

Fig. 2: Evolution with time of the proper size of a density perturbation λ and of the Hubble radius R_H, showing "horizon entry" at time t_{enter}.

and

$$\rho_r = \frac{3}{32\pi G t^2} \tag{3.52}$$

then, at t_{eq},

$$M_H(t_{eq}) = \frac{c^3}{4G} t_{eq} \simeq 5 \times 10^{14} (\Omega_{m0} h^2)^{-2} \mathrm{M}_\odot \tag{3.53}$$

3.6.2 The Growth of superhorizon perturbations

To calculate the evolution of perturbations with $\lambda > R_H$, we need GR. However, we can take a short-cut that gives the correct answer. Regard the perturbation as a slightly closed (positively curved) universe superposed on a flat ($k = 0$) universe with the same expansion rate. The Friedman equation (2.6) gives

$$H^2 = \frac{8\pi}{3} G \rho_0 \quad (k = 0) \tag{3.54}$$

where the subscript 0 denotes the $k = 0$ case (not $t = t_0$!). Now consider a Friedmann model with the same expansion rate, H, but higher density $\rho = \rho_1$, so that it is positively curved:

$$H^2 = \frac{8\pi}{3} G\rho_1 - \frac{k}{a_1^2} \tag{3.55}$$

Let's compare these two (when the expansion rate is equal: the "Hubble flow condition"). The density contrast is then

$$\delta \equiv \frac{\rho_1 - \rho_0}{\rho_0} = \frac{k/a_1^2}{8\pi \mathrm{G}\rho_0/3} \tag{3.56}$$

If $\delta << 1$, then a_1 and a_0 differ only by a small amount and we can set $a_1 \simeq a_0 \equiv a$. Since $\rho_0 \propto a^{-4}$ in the radiation-dominated epoch while $\rho_0 \propto a^{-3}$ in the matter-dominated era, we find that fluctuations outside the horizon grow as:

$$\delta \propto \frac{a^{-2}}{\rho} \propto \begin{cases} a^2 & \text{for } t < t_{\rm eq} \\ a & \text{for } t > t_{\rm eq} \end{cases} \tag{3.57}$$

4. THE EVOLUTION OF INDIVIDUAL PERTURBATIONS

4.1 Adiabatic and isothermal perturbations

The radiation energy density is $\rho_r c^2 = \sigma_r T^4$, where σ_r is the radiation constant and the radiation pressure is $P_r = \rho_r c^2/3$. Thus, the entropy of radiation per unit volume is given by

$$S_r = \frac{\rho_r c^2 + P_r}{T} = \frac{4}{3}\frac{\rho_r c^2}{T} = \frac{4}{3}\sigma_r T^3 \tag{4.1}$$

The entropy per unit mass is therefore

$$S \propto \frac{T^3}{\rho_m} \propto \frac{\rho_r^{3/4}}{\rho_m} \tag{4.2}$$

There are two basic types of perturbations: adiabatic or curvature perturbations and isothermal or entropic perturbations. A perturbation that leaves S invariant is called *adiabatic*. It consists of perturbations in both the matter density, ρ_m and the radiation density, ρ_r (or equivalently, the radiation temperature T). Thus:

$$\frac{\delta S}{S} = \frac{3}{4}\frac{\delta\rho_r}{\rho_r} - \frac{\delta\rho_m}{\rho_m} = 3\frac{\delta T}{T} - \frac{\delta\rho_m}{\rho_m} = 0$$

so

$$\delta_m \equiv \frac{\delta\rho_m}{\rho_m} = 3\frac{\delta T}{T} = \frac{3}{4}\frac{\delta\rho_r}{\rho_r} \equiv \frac{3}{4}\delta_r \tag{4.3}$$

A perturbation in the matter component only, $\delta_m \neq 0$, keeping the radiation component uniform is called an *entropic* or *isothermal* perturbation. (These are related to *isocurvature* fluctuations). During the radiation era, isothermal fluctuations in the baryons do not grow because of the strong frictional drag between matter and the uniform radiation field. After recombination, perturbations in the matter evolve in the same way regardless of whether they are adiabatic or isothermal initially. If perturbations are produced by "microscopic" physics, then we expect them to be adiabatic.

4.2 Perturbations in the dark matter

Consider the evolution of a perturbation in the dark matter (DM), of proper wavelength $\lambda \propto a$, which enters the horizon at $a = a_{enter}$. DM particles can be relativistic at early times; their velocity dispersion, $v \simeq c$ for $a < a_{nr}$ (where a_{nr} is the expansion factor at which the particles cease to be relativistic.) For $a > a_{nr}$, the velocity decays as $v \propto a^{-1}$ (because the DM behaves like an adiabatically expanding fluid: the entropy $S \propto T/\rho^{2/3} \simeq \mathrm{const} \Rightarrow T \propto v^2 \propto \rho^{2/3} \propto a^{-2} \Rightarrow v \propto a^{-1}$. Alternatively: $PV^{5/3} = \mathrm{const} \Rightarrow \rho T a^5 \propto T a^2 = \mathrm{const} \Rightarrow T \propto v^2 \propto a^{-2}$)

We can calculate the Jeans length and mass for the dark matter from eqns (3.34) and (3.35). Recall that $\overline{\rho}$ in these equations refers to the gravitationally dominant component, ρ_{tot}. For $a < a_{eq}$, $\rho_{\mathrm{tot}} = \rho_r \propto a^{-4}$; for $a > a_{eq}$, $\rho_{\mathrm{tot}} = \rho_m \propto a^{-3}$. Also, for dark matter v plays the role of the sound speed, $c_s \simeq v$. Thus, the Jeans length (3.34) is:

$$\lambda_J \propto \frac{v}{\overline{\rho}^{1/2}} \propto \begin{cases} a^2 & \text{for } a < a_{\mathrm{nr}} \\ a & \text{for } a_{\mathrm{nr}} < a < a_{\mathrm{eq}} \\ a^{1/2} & \text{for } a_{\mathrm{eq}} < a \end{cases} \tag{4.4}$$

If a perturbation enters the horizon when the dark matter is still relativistic, then that perturbation is quickly damped because of *free streaming*: the particles which are moving at relativistic speeds in 3D simply move out of the high density part of the perturbation which is then erased.

The most relevant perturbations are those that enter the horizon between $a_{\rm nr}$ and $a_{\rm eq}$. There are 3 stages in the evolution of such a perturbation.

<u>STAGE A</u>: $a < a_{\rm enter}$

Here $\lambda > \lambda_{\rm H}$, so perturbation grows. From (3.57),

$$\delta \propto a^2 \tag{4.5}$$

<u>STAGE B</u>: $a_{\rm enter} < a < a_{\rm eq}$

Now $\lambda < \lambda_{\rm H}$. However, the perturbation cannot grow because of the Mesźaros effect, so

$$\delta = \mathrm{const} \tag{4.6}$$

<u>STAGE C</u>: $a_{\rm eq} < a$

Still $\lambda < \lambda_{\rm H}$. If $\lambda > \lambda_J$, the perturbation can grow, so from (3.27),

$$\delta \propto a \tag{4.7}$$

We can also express the evolution in terms of the Jeans mass:

$$M_J \propto \rho_{\rm DM}\lambda_J^3 \propto \begin{cases} a^3 & \text{for } a < a_{\rm nr} \\ \mathrm{const} & \text{for } a_{\rm nr} < a < a_{\rm eq} \\ a^{-3/2} & \text{for } a_{\rm eq} < a \end{cases} \tag{4.8}$$

which can be compared with the horizon dark matter mass:

$$M_{\rm H} \propto \rho_{\rm DM} d_{\rm H}^3 \propto \begin{cases} a^2 & \text{for } a < a_{\rm nr} \\ a^3 & \text{for } a_{\rm nr} < a < a_{\rm eq} \\ a^{3/2} & \text{for } a_{\rm eq} < a \end{cases} \tag{4.9}$$

Fig. 3: Evolution with expansion factor a of the Jeans mass for the dark matter.

From these scalings, we can compute $M_J(z)$ using the definition of (3.35). For example, for cold dark matter, $M_J \simeq 10^6(\Omega_{m0}h^2)^{-2}(a/a_{eq})^{-3/2}\mathrm{M}_\odot$ for $a > a_{eq}$.

4.3 Perturbations in the baryons

Let a_{br} be the epoch of baryon-radiation equality, ie when $\rho_b = \rho_r$. This is given by:

$$1 + z_{\mathrm{br}} = \frac{\rho_b^0}{\rho_r^0} = 3.9 \times 10^4(\Omega_{b0}h^2) \tag{4.10}$$

Since $z_{\mathrm{rec}} \simeq 1100$, $z_{\mathrm{br}} > z_{\mathrm{rec}}$ only if $\Omega_{b0}h^2 > 0.026$. Current data indicate that $\Omega_{b0}h^2 \simeq (0.023 \pm 0.002)$ so, most likely, $z_{\mathrm{br}} < z_{\mathrm{rec}}$. However, since the two redshifts are so close, we will make the approximation $z_{\mathrm{br}} \simeq z_{\mathrm{rec}}$

For $a < a_{\mathrm{rec}}$, we have $\rho = \rho_b + \rho_r + \rho_{\mathrm{DM}}$ and $P =$

$P_r + P_b \simeq P_r = \frac{1}{3}\rho_r c^2$. Thus, (assuming $\rho_b/\rho_r << 1$), the adiabatic sound speed for the baryons is:

$$c_s^{(a)} = \left(\frac{\partial P}{\partial \rho}\right)_S^{1/2} = \frac{c}{\sqrt{3}} \qquad (4.11)$$

because the baryons are coupled to the radiation that contributes the pressure. Thus, we have the following 2 regimes:

$$c_s^{(a)} = \begin{cases} \frac{c}{\sqrt{3}} & = 1.7 \times 10^8 \text{ ms}^{-1} & \text{for } a < a_{\rm rec} \\ \left(\frac{\gamma kT}{m_p}\right)^{1/2} & = 5 \times 10^5 \left(\frac{1+z}{1+z_{\rm rec}}\right)^{1/2} \text{ms}^{-1} & \text{for } a_{\rm rec} < a \end{cases} \qquad (4.12)$$

where we have assumed $\gamma = 5/3$ and $T \simeq T_{rec} = 4000^o$K. (In reality, $T_b \simeq T_r$ only for $z \geq 300$; after that and until the universe is reionized by stars and quasars, $T_b \propto (1+z)^2$.) The sound speed plummets from $\sim 10^8 \text{ms}^{-1}$ before recombination to $\sim 5 \times 10^5 \text{ms}^{-1}$ just after recombination, as the radiation pressure ceases to act on the gas and the baryons can experience only gas pressure.

Following the same reasoning as for DM, we find:

$$M_{Jb} \propto \rho_b \lambda_J^3 \propto \begin{cases} a^3 & \text{for } a < a_{\rm eq} \\ a^{3/2} & \text{for } a_{\rm eq} < a < a_{\rm rec} \\ \text{const} & \text{for } a_{\rm rec} < a \end{cases} \qquad (4.13)$$

Thus, for example, from (4.12) and (3.35),

$$M_{Jb} = 3.2 \times 10^{14} \left(\frac{\Omega_{b0}}{\Omega_{m0}}\right) (\Omega_{m0} h^2)^{-2} \left(\frac{a}{a_{\rm eq}}\right)^3 M_\odot \qquad \text{for } a < a_{\rm eq} \qquad (4.14)$$

Fig. 4: Evolution with expansion factor a of the Jeans mass for the baryons.

and, directly from (4.12),

$$M_{Jb} = 5\times10^4 \left(\frac{\Omega_{b0}}{\Omega_{m0}}\right) (\Omega_{m0} h^2)^{-1/2} M_\odot \simeq \text{constant} \qquad \text{for } a_{\text{rec}} < a \tag{4.15}$$

(M_{Jb} remains constant as long as $T_b \simeq T_r$; thereafter the behaviour is roughly proportional to $(1+z)^{3/2}$). An adiabatic perturbation with $\lambda > \lambda_{Jb}(a_{eq})$ behaves just like a perturbation in the dark matter:

$$\delta_b \equiv \left(\frac{\delta\rho}{\rho}\right)_b \propto \begin{cases} a^2 & \text{for } a < a_{\text{enter}} \\ \text{const} & \text{for } a_{\text{enter}} < a < a_{eq} \\ a & \text{for } a_{eq} < a \end{cases} \tag{4.16}$$

Adiabatic perturbations $\lambda < \lambda_{Jb}(\text{rec})$ during the epoch $a_{\text{enter}} < a < a_{\text{rec}}$ oscillate as acoustic waves with sound speed $c_s^{(a)}$ and constant amplitude. During these oscillations, photons have enough time to diffuse out of fluctuations with $M < M_D(t) << M_{Jb}$ where

$$M_D \simeq 8\times10^7 \left(\frac{\Omega_{b0}}{\Omega_{m0}}\right)^{3/2} (\Omega_{m0}h^2)^{-5/4} \left(\frac{1+z}{1+z_{\rm eq}}\right)^{-15/4} M_\odot, \tag{4.17}$$

so no baryon fluctuations below this mass survive. At recombination,

$$M_D(z_{\rm rec}) \simeq 6.2 \times 10^{12} \left(\frac{\Omega_{b0}}{\Omega_{m0}}\right)^{3/2} (\Omega_{m0}h^2)^{-5/4} M_\odot, \tag{4.18}$$

Note:

• Baryon perturbations with $\lambda < \lambda_{Jb}$ can only grow after $a_{\rm rec}$, but perturbations in the dark matter can grow from $a_{\rm eq}$. During this time, perturbations in the DM grow by a factor $(a_{\rm rec}/a_{\rm eq}) \simeq (T_{\rm eq}/T_{\rm rec}) \simeq 21\Omega_{m0}h^2$. When baryons decouple at recombination, they quickly "fall into" the potential wells created by the DM. Thus, δ_b grows rapidly for a short time after $a_{\rm rec}$ until it catches up with $\delta_{\rm DM}$. Thereafter both grow as a.

• We have assumed $\Omega = 1$. If $\Omega \neq 1$, only the period $a > a_{\rm rec}$ is affected. As we saw in §3.3, when $\Omega \neq 1$, fluctuations eventually stop growing when the universe begins to expand too fast. One can show that growth stops when $\Omega_{m0}z \simeq 1$ ie for $z \leq \frac{1}{\Omega_{m0}}$

5. THE POWER SPECTRUM OF DENSITY FLUCTUATIONS

In this section, we collect the results of the preceeding section to arrive at a simple statistical description of the density and velocity fluctuation fields.

5.1 Mathematical background

The density field, $\delta(\overline{x}, t)$, can be written as a sum of plan waves in Fourier space:

$$\delta(\overline{x}, t) = \frac{1}{(2\pi)^3} \int \delta_k(\overline{k}, t) e^{i\overline{k}\cdot\overline{x}} d^3k$$

$$\delta_k(\overline{k}, t) = \int \delta(\overline{x}, t) e^{-i\overline{k}\cdot\overline{x}} d^3x \tag{5.1}$$

It is easy to see that one integral is the inverse of the other by using the definition of the Dirac delta, δ_D,

$$\int e^{i(\overline{k}-\overline{k}')\cdot\overline{x}} d^3x = (2\pi)^3 \delta_D(\overline{k} - \overline{k}') \tag{5.2}$$

where

$$\int \delta_D(\overline{x}) d^3x = 1 \tag{5.3}$$

and

$$\int f(\overline{x}) \delta_D(\overline{x} - \overline{x}')) d^3\overline{x} = f(\overline{x}') \tag{5.4}$$

Using these relations, it is straightforward to show that:

$$\int\int f(\overline{k}) e^{i\overline{k}\cdot\overline{x}} d^3x d^3k = (2\pi)^3 f(0) \tag{5.5}$$

The fact that $\delta(\overline{x}, t)$ averages to zero, $< \delta(\overline{x}, t) >= 0$, implies that $\delta_k(0) = 0$.

δ_k is usually assumed to be a *Gaussian random field* which means that the waves have random phases. In this case, the field can be specified entirely by its variance or power spectrum:

$$P(\overline{k}) \equiv |\delta_k(\overline{k})|^2 \tag{5.6}$$

For an isotropic distribution, the power spectrum, averaged over all possible realizations, must be independent of direction, $< P(\overline{k}) >= P(k)$. Often, $P(k)$ is approximated as

$$P(k) \propto k^n \tag{5.7}$$

In practice, $P(k)$ is not a power-law, ie n varies with scale.

The density and mass fluctuation within a region of volume V is defined as:

$$\frac{\delta\rho}{\rho} = \frac{\delta M}{M} = \frac{1}{V}\int \delta(\overline{x})d^3x \tag{5.8}$$

If δ_k is Gaussian, so is $\delta M/M$. The average density fluctuation (averaged over the whole of space) is zero:

$$\left\langle \frac{\delta M}{M} \right\rangle = 0 \tag{5.9}$$

It is (relatively) easy to find the mean squared value of $\delta M/M$:

$$\sigma^2(M) = \left\langle \left(\frac{\delta M}{M}\right)^2 \right\rangle = \frac{1}{(2\pi)^3 V_u}\int P(k)W(\overline{k};V)d^3k \tag{5.10}$$

where W is the *window function*:

$$W(\overline{k};V) = \left(\frac{1}{V}\int_V e^{i\overline{k}\cdot\overline{x}}d^3x\right)^2 \tag{5.11}$$

(Those who wish to derive (5.10), need to use eqn. (5.2) after multipling and dividing the integrand of $\sigma^2(M)$ by $e^{i\overline{k}\cdot\overline{x}}e^{i\overline{k}'\cdot\overline{x}_0}$, where x_0 is the variable over which the spatial average is being taken.)

For a spherical volume of radius x, it can be shown that:

$$W = \frac{9}{(kx)^6}[\sin(kx) - kx\cos(kx)]^2 \tag{5.12}$$

The exact form of the window function is not crucial. For a finite volume, W always tends to one at small k (large scales; because wavelengths much larger than the smoothing length contribute in full) and tends to zero at large k (small scales; because wavelengths much smaller than the smoothing length do not contribute). Often W is approximated by a sharp cut-off in k-space:

$$W(kx) = \begin{cases} 1 & \text{for } kx \leq 1 \\ 0 & \text{for } kx > 1 \end{cases} \tag{5.13}$$

With this approximation, and assuming $P(k)$ as in (5.7):

$$\sigma^2 \propto \int_0^{\frac{1}{x}} k^n d^3k \propto \left(\frac{1}{x}\right)^{3+n} \propto M^{-\left(\frac{3+n}{3}\right)} \tag{5.14}$$

since $M \propto x^3$, provided $n > -3$. We will express the present-day value of σ, assuming linear evolution, as:

$$\sigma \equiv \sigma_0 \left(\frac{M}{M_0}\right)^{-\alpha} \qquad \alpha = \frac{3+n}{6} \tag{5.15}$$

This gives the relative amplitude of fluctuations on different scales according to linear theory. Of course, σ scales with time in the same way as δ, ie as in (4.14):

$$\sigma = \sigma_0 g a \tag{5.16}$$

5.2 Dark matter power spectra

Density perturbations are thought to be generated by quantum fluctuations during inflation. These are adiabatic, Gaussian perturbations. As the universe inflates, the perturbation is stretched beyond the horizon. Once inflation ends, however, perturbations eventually re-enter the horizon. By self-similiarity arguments, when they do so, they all have the *same amplitude*, independently of wavelength (or mass). Smaller fluctuations re-enter the horizon first.

Let us compute the shape of the power spectrum at some time, say, t_{rec}. Consider a fluctuation that enters the horizon in the matter-dominated regime. Since $\sigma \propto \delta$, $\sigma_H/\sigma_{rec} = a_H/a_{rec}$, where the subscript H refers to horizon crossing. Thus,

$$\sigma_{rec} = \sigma_H \frac{a_{rec}}{a_H} = \sigma_H \left(\frac{t_{rec}}{t_H}\right)^{2/3} \tag{5.17}$$

where σ_H =const (ie independent of mass). During this time, the horizon mass scales as

$$M_H \propto \overline{\rho}_H (ct_H)^3 \propto t_H \tag{5.18}$$

because $\overline{\rho} \propto t^{-2}$ in the matter-dominated regime. Since at a_{enter}, $M \simeq M_H$,

$$\sigma_{rec} \propto t_H^{-2/3} \propto M^{-2/3} \tag{5.19}$$

Comparing with (5.15), $\frac{3+n}{6} = \frac{2}{3}$, so

$$\alpha = \frac{2}{3} \qquad \text{or } n = 1 \tag{5.20}$$

This is called the Harrison-Zeldovich spectrum, $P(k) \propto k$.

Density fluctuations cannot grow during the radiation-dominated era and so the power spectrum is maintained at the value at horizon crossing. Thus $\sigma \rightarrow \mathrm{const}$ and, from (5.16), $\alpha \rightarrow 0$, so $n \rightarrow -3$. It is usual to write

$$P(k) = T^2(k) k \tag{5.21}$$

where $T(k)$ is the *transfer function*.

For **cold dark matter** (CDM), a fitting formula is:

$$T_{\mathrm{CDM}}(k) = \frac{\ln(1+2.34q)}{2.34q}[1+3.89q+(16.1q)^2+(5.46q)^3+(6.71q)^4]^{-1/4} \tag{5.22}$$

where

$$q = \frac{k}{\Gamma}h^{-1}\mathrm{Mpc} \tag{5.23}$$

and the *shape parameter* Γ:

$$\Gamma = \Omega_{m0}h \tag{5.24}$$

Hot dark matter (HDM) corresponds to the case where the dark matter particles are still relativistic at matter-radiation equality. In this case, the dark matter particles free-stream out of the perturbations as soon as they come into the horizon and so fluctuations are erased until such time as the particles cease to be relativistic. The cut-off wavelength is,

$$\lambda_{\mathrm{FS}} \propto m_X^{-1} \tag{5.25}$$

where m_X is the mass of the hot dark matter particle. A fitting formula for HDM is:

$$\Gamma_{\mathrm{HDM}}(k) \simeq e^{-3/9q-2.1q^2} \tag{5.26}$$

The power spectrum (or, more precisely, $k^3|\delta_k|^2$) for CDM and HDM is shown in Figure 5.

Accurate transfer functions may be obtained using the programme on

http://www/physics.nyu.edu/matiasz/CMBFAST/cmbfast.html

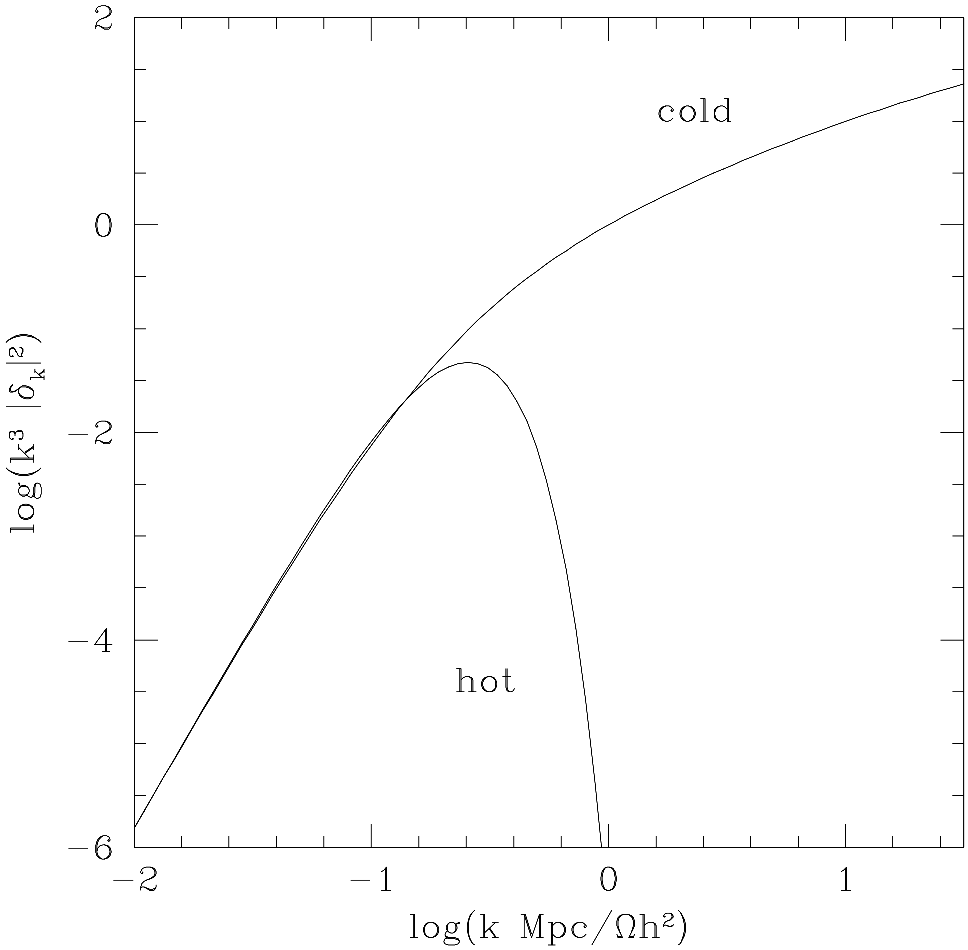

Fig. 5: The linear power spectra for cold dark matter and hot dark matter.

The discovery of fluctuations in the CMB by COBE in 1992 is one of the most important discoveries in XX century science. These fluctuations are a manifestation of the density fluctuations that give rise to galaxies. They connect the physics of the very early universe to the universe of galaxies and give us information about events during inflation and about the nature of the dark matter. In this section, we will explain the origin and significance of the recent data obtained by the WMAP satellite.

6.1 The angular power spectrum

The CMB temperature distribution on the celestial sphere can be expanded in spherical harmonics:

$$\frac{T(\theta,\phi)-T_0}{T_0} \equiv \frac{\Delta T(\theta,\phi)}{T} = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=+l} a_{lm}Y_{lm}(\theta,\phi) \tag{6.1}$$

where T_0 is the mean CMB temperature. This is analogous to the Fourier expansion of δ.

The $l = 0$ mode (monopole) gives the mean temperature; $l = 1$ (dipole) is mostly due to the motion of our galaxy in ($\simeq 600$km/s) produced by local mass fluctuations (Coma cluster, Great Attractor, etc); $l = 2$ (quadrupole) and higher modes are produced by intrinsic anisotropies produced either at t_{rec} (primary) or between t_{rec} and t_0 (secondary).

The multipole l is related to the angle on the sky by:

$$\theta \simeq \frac{60^0}{l} \tag{6.2}$$

for $l > 0$.

The angular power spectrum (analogous to $P(k)$) is:

$$C_l \equiv < |a_{lm}|^2 > \tag{6.3}$$

The autocorrelation function is defined as:

$$C(\theta) = < \frac{\Delta T}{T}(\hat{n}_1) \frac{\Delta T}{T}(\hat{n}_2) > \tag{6.4}$$

where $\cos\theta = \hat{n}_1 \cdot \hat{n}_2$ and the average is over the ensemble of possible skies. Since we have only one sky, we use the *ergodic* hypothesis: we can replace an average over realizations with an average over different patches of a single realization. (But this is not possible on large scales, where measurements are limited by *cosmic variance*.) Using the addition theorem for spherical harmonics,

$$P_l(\cos\theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{\infty} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi') \tag{6.5}$$

where P_l is the Legendre polynomial, it is easy to show that:

$$C(\theta) = \frac{1}{4\pi} \sum_{l=2}^{\infty} (2l+1) C_l P_l(\cos\theta) \tag{6.6}$$

6.2 Primary anisotropies

There are 3 basic effects occuring at the surface of last scattering:

1. Gravitational (Sachs-Wolfe) perturbations: photons from high density regions at last scattering have to climb out of potential wells and are thus redshifted.

2. Intrinsic (adiabatic) perturbations: in high baryon density regions, the coupling of matter and radiation compresses the radiation also, giving rise to high T.

3. Velocity (Doppler) perturbations: the velocity of the plasma at recombination produces Doppler shifts in frequency and thus in temperature.

A full treatment requires solving the Boltzmann eqn. (which is what CMBFAST does).

6.21 The Sachs-Wolfe effect

This is the dominant contribution to $\Delta T/T$ on large scales. Density fluctuations generate fluctuations in the gravitational potential, $\delta\phi$. Photons emitted from a potential well need to climb out and are subject to (a) gravitational redshift and (b) time dilation (we see photons at different times or values of a). To calculate these exactly we need GR, but we can get an approximate answer. Firstly, note that:

$$\delta\phi \simeq \delta\left(\frac{GM}{r}\right) = -\frac{GM}{r^2}\delta r = -\frac{GM}{r}\left(\frac{\delta r}{r}\right) \quad (6.7)$$

and also that $\phi \simeq GM/r \sim c^2$ when a perturbation comes inside the horizon. The gravitational redshift causes a change of wavelength λ. Since the radiation is very nearly black body, $\lambda \propto 1/T$. Thus,

(a) $\Rightarrow$

$$\left(\frac{\Delta T}{T}\right) = -\frac{\delta\lambda}{\lambda} = -\frac{\delta r}{r} = \frac{\delta\phi}{c^2} \quad (6.8)$$

The time dilation is due to the fact that photons emerging from a fluctuation are delayed relative to photons in an unperturbed region. Thus, using

$$\left(\frac{\Delta T}{T}\right) = \frac{\delta a}{a} = \frac{2}{3}\frac{\delta t}{t} \quad (6.9)$$

Since $\frac{\delta t}{t} = \frac{\delta r}{r} = -\frac{\delta\phi}{c^2}$, then

(b) $\Rightarrow$

$$\left(\frac{\Delta T}{T}\right) = -\frac{2}{3}\frac{\delta\phi}{c^2} \tag{6.10}$$

so, the net effect is:

$$\left(\frac{\Delta T}{T}\right) = \frac{1}{3}\frac{\delta\phi}{c^2} \tag{6.11}$$

6.22 Adiabatic perturbations

For adiabatic fluctuations, eqn (4.3) $\Rightarrow$

$$\frac{\Delta T}{T} = \frac{1}{3}\frac{\delta\rho}{\rho} \tag{6.12}$$

6.23 Doppler perturbations

Density fluctuations induce streaming motions. This produces a ΔT because some electrons are moving towards the observer when they last scatter the radiation and others are moving away. From the continuity eqn., $\delta\rho/t \sim \rho\nabla\cdot\overline{v} \sim \rho v/\lambda$, so

$$\frac{\Delta T}{T} \simeq -\frac{\Delta\lambda}{\lambda} = -\frac{v}{c} \simeq \frac{\delta\rho}{\rho}\left(\frac{\lambda}{ct}\right) \tag{6.13}$$

The Sachs-Wolfe effect dominates for scales ≥ 1 Gpc (ie larger than the horizon at t_{rec}); Doppler effects take over for scales ≤ 1Gpc, but are soon dominated by adiabatic effects on the smallest scales.

The horizon scale at t_{rec} corresponds to $l \simeq 100-200$. On these scales an *acoustic* peak is produced. This was first detected from ballon-borne measurements and has now been confirmed by the WMAP data. The peak results from the acoustic oscillations of sub-Jeans mass fluctuations in the photon-baryon fluid prior to t_{rec}. These standing waves have a phase

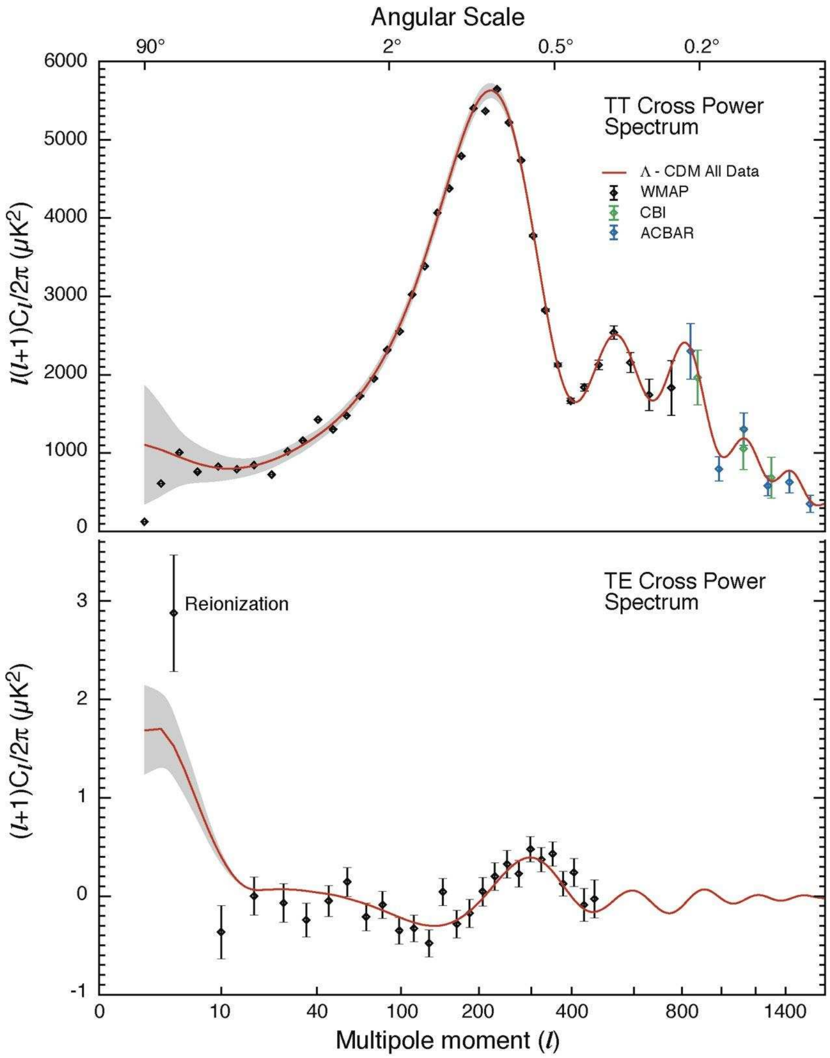

Fig. 6: The CMB temperature power spectrum measured by WMAP. The top plot shows the T-T correlation and the bottom plot the T-P correlation, where P is the polarization.

relation between density and velocity. When the Universe recombines, waves interfere constructively giving rise to the peak and its harmonics.

The physical lengthscale at which the first peak occurs corresponds to the size of the *sound horizon*, $R_{SH} \simeq c_s t_{rec}$ at the surface of last scattering. This is approximately fixed in all cosmological models. However, the angular scale subtended by R_{SH} depends on cosmology via the angular diameter-distance relation. For a flat universe, the peak occurs at $l \simeq 200$. For a positively curved universe, the angle subtended is larger, so the peak appears at smaller l. For a negatively curved universe, the angle subtended is smaller, so the peak appears at larger l. Thus, the position of the peak gives an estimate of Ω_{tot}. WMAP data give $\Omega_{tot} = 1.00 \pm 0.02$. The properties of the various peaks depend on other cosmological parameters in a complicated fashion. The C_ls measured by WMAP can be used to estimate the values of these parameters. Remarkably, the WMAP data agree very well with predictions of the CDM theory made in the early 1980s!

Fig. 7: Effect of spatial curvature on angular size of sound horizon at recombination.

The general non-linear solution for the evolution of density perturbations is complicated. However, we can obtain a simple solution for a spherically symmetric system. We will derive this solution and then use it to estimate the collapse times of objects from the linear density field. We will assume that we are in the matter-dominated regime which is the regime relevant to galaxy formation. However, the arguments can be generalized to include radiation and/or a cosmological constant.

7.1 The spherical top-hat

For a spherically symmetric distribution of matter, Birkhoff's theorem from General Relativity says that the gravitational attraction on a test particle at radius r is Gm/r^2 where m is the matter contained within that radius - the attraction of matter outside that radius averages to zero.

The equation of motion is therefore

$$\ddot{r} = -\frac{Gm}{r^2}. \tag{7.1}$$

This equation has parametric solution

$$\begin{aligned} r &= r_t \sin^2\theta \\ t &= \left(\frac{r_t^3}{2Gm}\right)^{\frac{1}{2}} (\theta - \frac{1}{2}\sin 2\theta). \end{aligned} \tag{7.2}$$

as may be verified by direct substitution. Here r_t is the maximum, or *turnaround* radius which occurs for

$\theta = \frac{\pi}{2}$, ie at a time

$$t_t = \frac{\pi}{2}\left(\frac{r_t^3}{2Gm}\right)^{\frac{1}{2}} = \left(\frac{3\pi}{32G\rho_t}\right)^{\frac{1}{2}}, \tag{7.3}$$

where ρ_t is the average density within radius r_t at turnaround ($\rho_t = M/(\frac{4}{3}\pi r^3) = \overline{\rho} + \delta\rho$).

Note that the relation between θ and t depends only upon ρ, so that if we start off with a uniform density perturbation, then it will remain uniform. This is the basis of the *top-hat* model. Note:

• If we surround the top-hat with vacuum out to the radius at which the mean density becomes equal to the background density, then the perturbation becomes invisible to the exterior universe - this is known as the *Swiss-cheese* model

• More realistically, the peak will be surrounded by matter which will fall onto it at late times - this is known as secondary infall.

• It is not necessary for ρ to be uniform in order for equations (7.2) and (7.3) to hold. For a typical *bowler-hat* profile, the central regions of a radially-declining density profile will collapse first.

7.2 Relation to linear theory

In linear theory, from equation (3.30), we have

$$\delta = \delta_0 g a \approx \delta_0 g_i \left(\frac{t}{t_0}\right)^{\frac{2}{3}}, \tag{7.4}$$

where $g \sim g_i \approx 1.25$ in the matter-dominated regime and t_0 is the current age of the Universe. We can rewrite this equation in terms of r by noting that

$$\rho r^3 = \frac{3m}{4\pi} = \text{constant}, \tag{7.5}$$

whence

$$\bar{\rho}_0 r_0^3 \;=\; (\bar{\rho} + \delta\rho) r^3 \;=\; \bar{\rho}(1\,+\,\delta) r^3, \qquad (7.6)$$

where $\bar{\rho}_0$ and r_0 refer to the density and radius that the spherical region would have had today if it had zero overdensity, ie r_0 is the radius of the sphere today from which matter at the mean density $\bar{\rho}_0$ would need to be swept up in order to produce the perturbation δ. Solving for r and keeping only linear terms gives

$$\begin{aligned} r \;&=\; r_0 \left(\frac{\bar{\rho}_0}{\bar{\rho}}\right)^{\frac{1}{3}} (1\,+\,\delta)^{-\frac{1}{3}} \\ &=\; r_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3}} \left[1 - \frac{1}{3}\delta_0 g_i \left(\frac{t}{t_0}\right)^{\frac{2}{3}}\right], \qquad (7.7) \end{aligned}$$

where we have used the fact that $\bar{\rho} \propto t^{-2}$ in the matter-dominated regime. (There should be a correction term for the deviation from matter-domination at late times, but this is unimportant for the argument that follows.)

Eqn (7.7) gives $r(t)$ to first order in linear theory. Equations (7.2) give the full solution in parametric form. The strategy is to match these two solutions in the linear regime in order to derive the conditions for collapse. We first expand the full solution of the spherical collapse problem, equation (7.2), for $\theta \ll 1$ to obtain the corresponding expression for the linear growth of the spherical top-hat.

$$\begin{aligned} \sin\theta \;&=\; \theta - \frac{1}{6}\theta^3 + \frac{\theta^5}{120}\cdots \\ \Rightarrow r \;&=\; r_t \theta^2 (1\,-\,\frac{1}{3}\theta^2\,+\cdots) \qquad (7.8) \\ t \;&=\; \left(\frac{r_t^3}{2Gm}\right)^{\frac{1}{2}} \left[\theta - \frac{1}{2}\left(2\theta - \frac{(2\theta)^3}{6} + \frac{(2\theta)^5}{120} - \cdots\right)\right] \end{aligned}$$

$$= \left(\frac{r_t^3}{2Gm}\right)^{\frac{1}{2}} \frac{2}{3}\theta^3(1 - \frac{1}{5}\theta^2 + \cdots)$$

$$\Rightarrow \theta^3 = \left(\frac{2Gm}{r_t^3}\right)^{\frac{1}{2}} \frac{3t}{2}(1 + \frac{1}{5}\theta^2 + \cdots) \tag{7.9}$$

Now

$$m = \frac{4\pi}{3}\bar{\rho}_0 r_0^3 = \frac{2r_0^3}{9Gt_0^2}, \tag{7.10}$$

because in the matter-dominated era,

$$\bar{\rho}_0 = \frac{3H_0^2}{8\pi G} = \frac{1}{6\pi G t_0^2}. \tag{7.11}$$

Substituting (7.10) in (7.9),

$$\theta^3 = \left(\frac{r_0}{r_t}\right)^{\frac{3}{2}} \left(\frac{t}{t_0}\right)(1 + \frac{1}{5}\theta^2 + \cdots) \tag{7.12}$$

and then substituting this back into (7.8) gives

$$\begin{aligned} r &= r_0\left(\frac{t}{t_0}\right)^{\frac{2}{3}}(1 - \frac{1}{3}\theta^2 + \cdots)(1 + \frac{1}{5}\theta^2 + \cdots)^{\frac{2}{3}} \\ &= r_0\left(\frac{t}{t_0}\right)^{\frac{2}{3}}(1 - \frac{1}{5}\theta^2 + \cdots) \\ r &= r_0\left(\frac{t}{t_0}\right)^{\frac{2}{3}}\left(1 - \frac{1}{5}\left(\frac{r_0}{r_t}\right)\left(\frac{t}{t_0}\right)^{\frac{2}{3}} + \cdots\right). \end{aligned} \tag{7.13}$$

Comparing (7.7) and (7.13), we see that to get the correct linear growth, we have to make the identification

$$\boxed{r_t = \frac{3}{5\delta_0 g_i} r_0} \tag{7.14}$$

7.3 Halo properties

Equation (7.14) is the basic relation used to calculate the properties of protogalactic halos. It relates the radius at turnaround to the radius and density normalized at some early time when the evolution was linear.

In practice, we use the current day (comoving) radius, r_0, of the sphere that contains the same mass as the perturbation but at the mean density, and the current overdensity, δ_0, assuming that the evolution were linear.

The time at maximum expansion, or turnaround, equation (7.3), can be expressed using (7.10) and (7.14) as

$$t_t = \frac{\pi}{2}\left(\frac{r_t^3}{2Gm}\right)^{1/2} = \frac{\pi}{2}\frac{3}{2}\left(\frac{r_t}{r_0}\right)^{3/2} t_0 = \frac{3\pi}{4}\left(\frac{3}{5\delta_0 gi}\right)^{3/2} t_0 \tag{7.15}$$

which, from eqn. (7.4) corresponds to a linear overdensity of

$$\delta_t = \delta_0 g_i \left(\frac{t_t}{t_0}\right)^{2/3} = \frac{3}{5}\left(\frac{3\pi}{4}\right)^{2/3} \simeq 1.06. \tag{7.16}$$

The actual density at this time is $\rho_t = m/r_t^3$ and the mean density is $\bar{\rho}_t = \frac{m}{r_0^3}\left(\frac{t}{t_0}\right)^{-2}$ so the density contrast is

$$\frac{\rho_t}{\bar{\rho}_t} = \left[\frac{r_0\left(\frac{t_t}{t_0}\right)^{2/3}}{r_t}\right]^3 = \left(\frac{3\pi}{4}\right)^2 \simeq 5.55 \tag{7.17}$$

At the time of maximum expansion, the kinetic energy of a top-hat density perturbation is zero. Denoting the kinetic energy by T and the potential energy by V, we have

$$T_t = 0 \qquad V_t = -\frac{3}{5}\frac{Gm^2}{r_t} \tag{7.18}$$

The collapse of the perturbation, if it is perfectly smooth, is a time-reversal of the expansion and therefore the collapse time is

$$t_c = 2t_t \tag{7.19}$$

corresponding to a linear overdensity

$$\delta_c = \delta_t \left(\frac{t_c}{t_t}\right)^{2/3} = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3} \simeq 1.69 \tag{7.20}$$

This result is often used when calculating the number density of collapsed halos given a particular linear density field.

In practice, the collapse will not be perfectly uniform and small irregularities will grow. The fluctuations in the gravitational potential mix up the particles and lead to virialization. This gives two equations relating the kinetic and potential energies of the collapsed object. Firstly, conservation of energy,

$$T_c + V_c = T_t + V_t = V_t \tag{7.21}$$

and secondly, the virial theorem,

$$2T_c + V_c = 0 \tag{7.22}$$

Together, these give

$$V_c = 2V_t \tag{7.23}$$

The usual assumption is that $V \,\alpha\, 1/r$, so using (7.14),

$$r_c = \frac{1}{2}r_t = \frac{3}{10\delta_0 g_i}r_0 \tag{7.24}$$

and

$$\frac{\rho_c}{\bar{\rho}_c} = \left[\frac{r_0\left(\frac{t_c}{t_0}\right)^{2/3}}{r_c}\right]^3 = 18\pi^2 \tag{7.25}$$

$\Rightarrow$

$$\boxed{\frac{\rho_c}{\bar{\rho}_c} \simeq 178} \tag{7.26}$$

Despite the naïvety of its derivation, (7.26) matches the observed density contrast of virialized systems in N-body simulations, as well as the inferred density contrast of galaxy and galaxy cluster halos.

We can now determine the mean properties of galactic halos. A halo of comoving linear overdensity δ_0 will collapse to form a virialised structure when its overdensity reaches δ_c. This occurs at an expansion factor

$$a_c = \frac{1}{1+z} = \frac{\delta_c}{\delta_0 g_i} \tag{7.27}$$

at which time the radius, mean density and velocity dispersion of the halo are

$$r_c = \frac{3}{10\delta_0 g_i} r_0 \simeq \frac{0.18}{1+z_c} r_0 \tag{7.28}$$

$$\rho_c = 18\pi^2 \bar{\rho}_c \simeq 180(1+z_c)^3 \bar{\rho}_0 \tag{7.29}$$

$$\sigma^2 = \frac{2T_c}{3m} = \frac{1}{5}\frac{Gm}{r_c} \simeq 2.8(1+z_c) G\bar{\rho}_0 r_0^2 \tag{7.30}$$

In each of these expressions we can convert from comoving size to mass using the relation 3 $m = 4\pi\bar{\rho}r_0^3$.

7.4 Scaling laws for galactic halos

Let us assume a power-law relationship between linear overdensity and mass for typical perturbations that will go on to form galaxies,

$$\delta_0 \propto 1 + z_c \propto m^{-\alpha} \tag{7.31}$$

where $\alpha > 0$. On galactic scales the observed value is $\alpha \approx \frac{1}{3}$. Then (since $r_0 \propto m^{1/3}$), equations (7.29) and (7.30) give the scaling laws:

$$r_c \propto \frac{r_0}{1+z_c} \propto m^{\alpha+1/3} \tag{7.32}$$

$$\rho_c \propto (1+z_c)^3 \propto m^{-3\alpha} \qquad (7.33)$$
$$\sigma^2 \propto r_0^2(1+z_c) \propto m^{2/3-\alpha} \qquad (7.34)$$

These scaling laws hold true for any self-similar scaling hierarchy even if we do not assume the top-hat model - only the constants of proportionality will change.

Equation (7.33) tells us that more massive galactic halos have lower mean densities (because they form later when the Universe has a lower density). If we assume a constant mass-luminosity ratio then the observations tell us that for elliptical galaxies $\rho \, \alpha \, m/r^3 \, \alpha \, m^{-1}$ which agrees with the expectation that $\alpha = \frac{1}{3}$,

Similarly, Equation (7.33) reproduces the Faber-Jackson relation, $L \, \alpha \, \sigma^4$, for a constant mass-luminosity ratio for $\alpha = \frac{1}{6}$, which is slightly lower than observed.

We can also establish a scaling relation that is independent of the mass-luminosity ratio. Eliminating m from (7.32) and (7.33),

$$r_c \propto \sigma^{\frac{2(3\alpha+1)}{2-3\alpha}}. \qquad (7.35)$$

Observations of elliptical galaxies indicate that $R_e \, \alpha \, \sigma^{8/3}$ which is consistent with the result above for $\alpha \approx 0.24$, slightly lower than $\frac{1}{3}$ but not far off.

Thus this simple model does a good job of explaining the basic properties of elliptical galaxies. Things do not work so well for spiral galaxies but that is not surprising because the luminosity of the disc is a poor indicator of the mass of the halo.

The above scaling laws give a single parameter family of galaxies (for each mass there is a unique radius) whereas we know that ellipticals form a two-parameter family that lie on the fundamental plane. This can be recovered by allowing a range of density fluctuations for a given mass, $\delta \, \alpha \, \nu \, m^{-\alpha}$.