2004-5
Galaxy Formation
Question 1

(a) Explain in words the physical signficance of (i) the Jean's Mass and (ii) the Meszaros effect. [5 marks]

(b) In the expanding universe, the first law of thermodynamics takes the form:

$$\frac{d\rho}{dt} + 3H\left(\rho + \frac{P}{c^2}\right) = 0$$

where ρ and P are the density and pressure, and H is the Hubble parameter. Show that for a universe dominated by relativistic particles or radiation, $\rho \propto a^{-4}$. [5 marks]

(c) Consider a universe dominated by the cosmological constant, Λ. The Friedmann equations may be written as:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{\Lambda c^2}{3}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_\Lambda + \frac{3P}{c^2}\right),$$

where a is the expansion factor, k the curvature of the universe, $\rho_\Lambda c^2$ the energy density associated with Λ and P the pressure. Writing the equation of state for Λ as $P = w\rho_\Lambda c^2$ and recalling that $\Lambda c^2 = 8\pi G\rho_\Lambda$, show that in a Λ-dominated universe $w = -1$. [5 marks]

(d) Estimate the Hubble radius at recombination in a flat, matter-dominated universe. How does the proper size of this region at the present day compare with today's Hubble radius? [5 marks]

(e) What is the dominant contribution to the temperature anisotropy on large scales? Recalling that photons emitted from a potential well at the epoch of recombination experience both a gravitational redshift and a time dilation, derive an expression for the Sachs-Wolfe effect. [5 marks]

Qu 1.   Solution to Question 1   2004-5

(a) (i) Jean's mass: mass for which the pressure force exactly balances the gravitational force of a density perturbation (3)

Mezaros effect: the drastic retardation in the growth of a collisionless, non-relativistic matter perturbation during the period when the Universe is radiation dominated (2)

bookwork

(b) For radiation, $p = \frac{1}{3}\rho c^2 \Rightarrow \frac{d\rho}{dt} + 3H\left(\frac{4}{3}\rho\right) = 0 \Rightarrow \frac{d\rho}{da}\dot{a} + 4\frac{\dot{a}}{a}\rho = 0$ (3)

$\Rightarrow \frac{d\rho}{\rho} = -4\frac{da}{a} \Rightarrow \underline{\underline{\rho \propto a^{-4}}}$ bookwork / unseen (2)

(c) From the 1st Friedmann eqn: $\dot{a}^2 + kc^2 = \frac{\Lambda c^2}{3}a^2$

$\Rightarrow 2\dot{a}\ddot{a} = 2a\frac{\Lambda c^2}{3}\dot{a} \Rightarrow \ddot{a} = \frac{\Lambda c^2}{3}a$ bookwork / unseen (3)

Thus, the 2nd Friedmann eqn $\Rightarrow \frac{\Lambda c^2}{3} = \frac{8\pi}{3}G\rho_\Lambda = -\frac{4\pi G}{3}(\rho_\Lambda + 3w\rho_\Lambda)$

$\Rightarrow 2 = -(1+3w) \Rightarrow w = -1$ (2)

(d) $R_H = \frac{c}{H}$; Now $H \propto \frac{1}{t}$ and $a \propto (1+z)^{-1} \propto t^{2/3} \Rightarrow R_H \propto (1+z)^{-3/2}$

$\therefore R_H(t_{rec}) = R_H(t_0)(1+z_{rec})^{-3/2} \simeq \frac{c}{H_0}(1000)^{-3/2} = \frac{3\times10^5 \text{ km/s}}{70 \text{ km/s}}\text{ Mpc } 3.2\times10^{-5}$, (2)

$\simeq \frac{10}{70}$ Mpc $= .14$ Mpc $\simeq 140$ kpc   bookwork / unseen

Today, this region has grown to $R = (1+z)R_H(t_{rec}) = 140$ Mpc (1)

The present day horizon is $R_H = \frac{3\cdot10^5}{70}$ Mpc $= 4.3\times10^3$ Mpc (2)

(e) Dominant contribution to $\Delta T/T$ on large scales is Sachs-Wolfe effect. bookwork (2) (2)

Gravitational redshift $\Rightarrow \left(\frac{\Delta T}{T}\right)_a = \frac{\delta\varphi}{c^2}$; time dilation $\Rightarrow \left(\frac{\Delta T}{T}\right)_b = -\frac{\delta a}{a} = -\frac{2}{3}\frac{\delta t}{t} = -\frac{2}{3}\frac{\delta\varphi}{c^2}$

because $a \propto t^{2/3}$ and, to 1st order, $\delta t/t \simeq \delta\varphi/c^2$.

Net effect is $\frac{\Delta T}{T} = \left(\frac{\Delta T}{T}\right)_a + \left(\frac{\Delta T}{T}\right)_b = \frac{\delta\varphi}{c^2} - \frac{2}{3}\frac{\delta\varphi}{c^2} = \frac{1}{3}\frac{\delta\varphi}{c^2}$ (1)