2004-5

Galaxy Formation

Question 2 (2 pages)

The equation that describes the linear growth of a density perturbation in a non-relativistic fluid of density ρ and pressure P in the matter era is

$$\ddot{\delta} + 2H\dot{\delta} = \frac{\nabla_r^2(\delta P)}{\overline{\rho}} + \nabla_r^2\phi,$$

where

$$\nabla_r^2\phi = 4\pi \mathrm{G}(\delta\rho_{\mathrm{tot}}).$$

Here, H is the Hubble parameter, $\overline{\rho}$ the mean density, $\delta \equiv \delta\rho/\overline{\rho}$ the density perturbation, ϕ the perturbation in the gravitational potential, (δP) the perturbation in the pressure and $(\delta\rho_{\mathrm{tot}})$ the perturbation in the *total* density.

(a) Show that for cold matter (dust), the perturbation equation reduces to

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\Omega_{\mathrm{m}}\delta = 0 \qquad (1)$$

where Ω_{m} is the matter density parameter. [2 marks]

Derive the growing and decaying mode solutions of equation (1) for a universe with critical matter density ($\Omega_m = 1$). [8 marks]

(b) Consider a universe dominated by a uniform dark energy field for which the Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{\Lambda c^2}{3},$$

where k is the curvature and Λ the cosmological constant. Show that density fluctuations in a flat universe of this kind remain frozen in at their initial value. [6 marks]

(c) Show that, whether primordial fluctuations are adiabatic or isothermal, the measured amplitude of temperature fluctuations in the microwave background radiation implies that collapsed structures such as galaxy clusters can only exist today if the dark matter is predominantly non-baryonic. [4 marks]

Question 2

Solution to Question 2 2004-5

(a) For cold matter, $P=0$ and $\delta P=0$

$\Rightarrow \ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\,\delta$. But $\Omega_m H^2 = \frac{8\pi}{3} G\bar{\rho}$ 1

$\Rightarrow \ddot{\delta} + 2H\dot{\delta} - \frac{3}{2} H^2 \Omega_m \delta = 0$ 1

bookwork

For $\Omega_m = 1$, $a \propto t^{2/3} \Rightarrow H = \frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t}$ 1

$\therefore \ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$

bookwork/unseen

Try $\delta = At^n \Rightarrow \dot{\delta} = Ant^{n-1}$ and $\ddot{\delta} = An(n-1)t^{n-2}$ 3

$\therefore An(n-1)t^{n-2} + \frac{4}{3}Ant^{n-2} - \frac{2}{3}At^{n-2} = 0$

$\Rightarrow n^2 + \frac{1}{3}n - \frac{2}{3} = 0 \Rightarrow n = \dfrac{-\frac{1}{3} \pm \sqrt{\frac{1}{9}+\frac{8}{3}}}{2} = \dfrac{-\frac{1}{3} \pm \sqrt{\frac{25}{9}}}{2} = \begin{cases} \frac{2}{3} \\ -1 \end{cases}$

4

$\therefore \delta = A(\bar{x})\,t^{2/3} + B(\bar{x})\,t^{-1}$, growing & decaying modes

(b) Flat $\Rightarrow k=0$, so we have $H^2 = \frac{\Lambda c^2}{3} = \text{const}$

unseen

The perturbation equation is now $\ddot{\delta} + 2H\dot{\delta} = 0$ 3

$\therefore \dfrac{d\dot{\delta}}{\dot{\delta}} = -2H\,dt \Rightarrow \dot{\delta} = c(\bar{x})\,e^{-2Ht}$

$\Rightarrow \delta = B(\bar{x})\,e^{-2Ht} + A(\bar{x})$ 3

(c) From CMB, $\frac{\delta T}{T} \sim 10^{-5}$. For adiabatic fluctuation $\delta_{bar} = \frac{3\delta T}{T}$; for isothermal fluctuation, $\delta_{bar} \lesssim \frac{\delta T}{T}$. Recombination occurred at $z \simeq 1000$. From (a), for $\Omega_{bar} = 1$, $\delta \propto a$, so fluctuation in the mass would have amplitude $\delta_{bar} \lesssim 10^3 \times 3 \times 10^{-5} \lesssim 3 \times 10^{-2} < 1$ 2

Thus, collapsed objects $(\delta \geq 1)$ could not have formed.

If $\Omega_{bar} < 1$, the growth factor since recombination is even smaller 2

bookwork / unseen