2004 - 5
Galaxy Formation
Question 3

Consider a uniform spherical overdense region of mass M and radius r in an Einstein-de-Sitter universe. A parametric solution of the equations of motion of this region is:

$$r = A(1 - \cos\eta) \qquad t = B(\eta - \sin\eta),$$

where A and B are constants.

(a) By considering the behaviour of this solution at early times ($\eta << 1$, $r \propto t^{2/3}$, mean cosmic density $\bar{\rho} \propto M/r^3$), show that

$$A^3 = GMB^2,$$

where G is Newton's gravitational constant. [8 marks]

(b) By taking into account that the mass of the overdense region is conserved, show that when the region turns around, its density is $9\pi^2/16$ times the critical density. [6 marks] If the region reaches maximum expansion at $t = t_{\rm ta}$, show that it collapses to a singularity at $t = 2t_{\rm ta}$. [2 marks]

(c) Assume that the age of the Universe is 10^{10} years and that the Coma cluster (of mass $10^{15} M_\odot$) has only just collapsed. Calculate the turnaround radius of Coma. [4 marks]

[$M_\odot = 2 \times 10^{30}$ kg; $G = 6.67 \times 10^{-11}$ Nm^{2}kg^{-2}; 1 pc $= 3.1 \times 10^{16}$m; 1 year $= 3.15 \times 10^7$s]

Qu 3

Solution to Question 3 2004-5

(a) At early times, $\eta \ll 1$ and $r \propto t^{2/3}$

$$\Rightarrow r = A\left(1-1+\frac{\eta^2}{2}\right) = \frac{A\eta^2}{2} \; ; \quad t = B\left(\eta - \eta + \frac{\eta^3}{6}\right) = \frac{B\eta^3}{6}$$ bookwork/unseen (2)

$$\frac{r}{r_0} = \left(\frac{t}{t_0}\right)^{2/3} \Rightarrow \frac{A\eta^2}{2r_0} = \left(\frac{B\eta^3}{6t_0}\right)^{2/3} \Rightarrow \frac{A}{2r_0} = \left(\frac{B}{6}\right)^{2/3} t_0^{-2/3} \quad \ldots(1)$$ (2)

But $t_0 = \frac{2}{3}\frac{1}{H_0}$ and $H_0^2 = \frac{8\pi}{3}G\rho_0 = \frac{8\pi}{3}G\frac{3M}{4\pi r_0^3} = \frac{2GM}{r_0^3} \Rightarrow t_0 = \frac{2}{3}\frac{r_0^{3/2}}{(2GM)^{1/2}}$ (2)

Subst in (1) $\Rightarrow \frac{A}{2r_0} = \left(\frac{B}{6}\right)^{2/3}\left(\frac{2}{3}\right)^{-2/3} r_0^{-1}(2GM)^{1/3} \Rightarrow A = B^{2/3}(GM)^{1/3} \Rightarrow \underline{\underline{A^3 = B^2 GM}}$ (2)

(b) At turnaround, $\eta = \pi \Rightarrow r_{tA} = A(1+1) = 2A$ and $t_{tA} = \pi B \quad \ldots(2)$ (2)

unseen

Mass conservation $\Rightarrow \rho = \rho_0\left(\frac{r}{r_0}\right)^{-3} \Rightarrow \rho = \rho_0\frac{r_0^3}{8A^3} = \frac{3M}{4\pi r_0^3}\frac{r_0^3}{8A^3} = \frac{3}{32}\frac{M}{\pi A^3}$ (2)

Now, $\rho_{crit} = \frac{3H^2}{8\pi G} = \frac{3}{8\pi G}\left(\frac{2}{3}\frac{1}{t_{tA}}\right)^2 = \frac{1}{6\pi^3 GB^2}$

$$\therefore \frac{\rho}{\rho_{crit}} = \frac{3M}{32\pi A^3}6\pi^3 GB^2 = \frac{9\pi^2}{16}\frac{M\,GB^2}{B^2GM} = \underline{\underline{\frac{9\pi^2}{16}}} \quad \ldots(3)$$ (2)

Now, $r \to 0 \Rightarrow \cos\eta = 1 \Rightarrow \eta = 2\pi \Rightarrow t = 2\pi B$

and (2) $\Rightarrow \underline{\underline{t = 2t_{tA}}}$ (2)

unseen

(c) $t_0 = 10^{10}$ yrs ; $t_c = t_0 = 2\pi B \Rightarrow B = \frac{t_0}{2\pi} \Rightarrow A = \left(\frac{t_0}{2\pi}\right)^{2/3}(GM)^{1/3}$

$$\Rightarrow r_{tA} = 2A = 2\left(\frac{t_0}{2\pi}\right)^{2/3}(GM)^{1/3} = \frac{2}{(2\pi)^{2/3}}\left[(10^{10})^{2/3}(4.4\times10^{-33}\times10^{15})^{1/3}\right]$$ (2)

$$= \frac{2}{(2\pi)^{2/3}}4.4^{1/3}\,10^{2/3}\text{ Mpc} = \underline{\underline{4.47\text{ Mpc}}}$$ (2)

$$\left[\; G = 6.67\times10^{-11}\frac{\text{kg m}}{s^2}m^2\,kg^{-2} = 6.67\times10^{-11}\frac{m^3}{s^2\,kg}\times\frac{1}{(3.1\times10^{22})^3}\frac{\text{Mpc}^3}{m^3}\times(3.15\times10^7)^2\frac{s^2}{yr^2}\times2\times10^{30}\frac{kg}{M_\odot}\right.$$

$$\left. = 4.4\times10^{-33}\frac{\text{Mpc}^3}{yr^2\,M_\odot}\;\right]$$