1 Introduction to general relativity

1.1 Admin

Recommended reading is Foster & Nightingale 'A Short Course in General Relativity', Springer. I also use Schutz 'A first course in General Relativity', CUP, but I also put all my lecture notes on the web. They are available in DUO or directly at http://duss0.dur.ac.uk/~done/foundations.html. For general reading (just for fun) try the Uncle Albert series by Russell Stannard, and Brian Greene 'The Elegant Universe'. These give you some nice ways to think about GR.

1.2 gravity=acceleration: Principle of equivalence

Newtons theory of gravity is amazingly accurate in calculating the effects of gravity in the world around us, but completely silent on what causes gravity. so what is gravity anyway? its obviously some form of acceleration. how is it different from pushing something? Einstens 'happiest thought' was the realisation that there was no difference between gravity and an accelerating frame - there is no difference between an accelerated frame without a gravitational field and a stationary frame within a gravitational field.

So there is a deep connection between gravity and accelerated motion. This explains why inertial and gravitational masses are the same, which was a big deep problem. Inertial mass, m_i, is the one which gives the constant of proportionality between force and acceleration - Newtons famous $F = m_i a$. Gravitational mass, m_g, determines how the object is affected by the force of gravity $F = GMm_g/r^2 = m_g g$. So when an object falls under a gravitational field we have $a = m_g/m_i g$. The only way for all objects to fall the same way in a gravitational field (feather and hammer) is if m_g/m_i = constant. If acceleration and gravity are two different things then this is deeply strange. Compare it with electrostatics - a test particle of charge q and inertial mass m_i in a field E experiences a force qE and acceleration $m_i a$. So $a = q/m_i E$ and different particles have difference values of q/m_i. But they don't have different values of m_g/m_i! But if gravity is the same as being accelerated then its not at all strange.

1.3 Acceleration

So there is this very deep connection between acceleration and gravity so to understand gravity we need to think about acceleration. Circular motion is a nice way to get an accelerated frame since the speed stays constant. So think of a spinning roundabout. One person crawls around ON THE GROUND with a ruler, and lays it end to end to measure the circumference. Another person crawls across to measure the diameter. The ratio circumference/diameter is then simply π.

But what does someone ON the roundabout measure ? If the ruler is small enough then the ruler lies along the direction of motion, so its length is shortened. so the person on the roundabout needs more rulers to get around the circumference than the person who crawled around the outside circumference who was at aproximately rest with respect to us. (yes, there are some sublties, but see the notes at the back of The Elegant Universe). But along the radius the ruler is perpendicular to the direction of motion so there is no length contraction so the diameter comes out to be the same as before. So the ratio circumference/diameter is BIGGER than π. There is NO WAY to do this in flat space! If we go onto the surface of a sphere (the surface falls away from us in both directions - positively curved space) then the ratio circumference/diameter is SMALLER than π (think of the limit of a circle of radius equal to that of the sphere - the circumference is $2\pi r$ but the diameter is πr so the ratio is 2!). If we go to the inside of a sphere we are still in positively curved space - it curves towards us in two directions. But we CAN do it in negatively curved space - like a saddle shape where it curves away from us in one direction and towards us the other.

So, if we are to handle acclerating frames we MUST use curved space. And since we know from special relativity that time is also a dimension we have to handle curved spacetime. So we have to do some fairly advanced maths in order to figure out path lengths in curved space - its called differential geometry.

1.4 Curved space

But how does this help us with understanding gravity? Think of ants with ink on their feet, walking normally i.e. taking steps of equal length with their left and right (and middle) feet. On a flat piece of paper their inky feet leave a path which is a straight line. On a curved surface, still taking steps of equal length, they trace out a curved path. e.g. two ants setting off from different points along the equator of a sphere, heading due south. Their paths get closer and closer together AS IF there were an attractive force between them. But there is not, its just that they are travelling normally (steps of equal length with left and right and middle feet) over a surface that is curved. Now put the ants on the surface of an apple. they still move 'normally' but now their paths show a variety of curves depending on the curvature of the surface of the apple - if the ants walk close to the stalk where the surface is highly curved then their path curves a lot.

so gravity = acceleration tells us ultimately that gravity = curvature. THERE IS NO FORCE. the 'walking normally' paths in physical language are constant velocity frames - i.e. inertial frames. But in geometric language they are geodesic paths - shortest distance between two points.

so there is no difference between a free fall frame in a gravitational field and an inertial frame. This is really the equivalence principle. *In a freely falling (nonrotating) frame then the laws of physics are the laws of special relativity.* THERE IS NO FORCE ACTING!!!!

locally, geodesics (shortest distance between two points) appear straight but over more extended regions of spacetime then geodesics originally receeding from each other begin to approach at a rate governed by the curvature of spacetime, and this effect of geometry on matter is what we mean today by the old word gravitation. and this warping of spacetime comes from matter/energy.

Space tells matter how to move, matter tells space how to curve

1.5 Implications for matter- an explanation of SR speed limit of c

mass curves spacetime so energy curves spacetime too as $E = mc^2$, mass and energy are equivalent. So any form of energy adds to curvature of spacetime which is gravity. so KE adds to the response of the particle to gravity i.e. adds to its mass. so when the KE gets to be a substantial fraction of the rest mass then additional acceleration will increase the response to gravity of the particle ie increase its mass, which stops the velocity increasing above c....!

1.6 Implications on light - lightbending

If gravity is curved spacetime, then everything that travels across spacetime is affected - so light is affected by gravity. This is not obvious at all in Newtonian gravity as Newtonian gravity affects things with mass $F = GMm/r^2$ - But $a = F/m = GM/r^2$. so we could argue either way in Newtonian gravity - that light has no mass so is not affected or that gravitational acceleration is independent of mass so affects everything. But by the beginning of the 20th century it was obvious that light speed was constant, so betting on gravity not affecting light seems like the way to go. But in Einsteins gravity, light is clearly affected as it travels across curved space time so its path will be curved. And this is what is SEEN in one of the first experimental tests of GR with light from distant stars which has a projected line of sight which lies close to the sun. These paths are curved, leading to a different apparent position than 6 months later when the star is over the other side of the sky from the sun (have to do the first measurement during a solar eclipse otherwise the starlight is drowned out by the sun. And the measured ammount agrees exactly with GR - which is 2x bigger than newtonian assuming GM/r^2 and definitely different from Newtonian straight line if light not affected by gravity

1.7 Implications for matter and light

We need gravity to affect light otherwise we could build an infinite energy machine!! This is the Pounds-Rebka-Snyder experiment. A particle dropped from a tower of height h has energy of its rest mass plus mgh at the bottom, then we converting this energy to a photon and send it back up the tower. if gravity doesn't affect light then it arives at the top with energy $h\nu = m_0c^2 + m_0gh$. And if we convert all this energy to mass then we get a particle of mass $m_1c^2 = m_0c^2 + m_0gh$ ie $m_1 > m_0$. Do this an infinite number of times and get infinite enrgy out!!! not a good plan. By contrast, if gravity can affect light then the photon loses the same amount of energy on the way up as the particle gained on the way down. this is gravitational redshift and its measureable and GR gets it right!

1.8 The way ahead

1) understand how to describe curvature
2) figure out how mass(energy) curves spacetime(!!!)
3) find the geodesic paths in our curved spacetime - these will be the natural 'straight line' paths that particles/photons will take if no forces acting

2 Mathematical Background

2.1 Curvature as distance: the metric

distance between points tells us the shape of the surface. and we've already seen how to do this in terms of a *metric*. e.g. special relativity has

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = \begin{pmatrix} cdt & dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}$$

or this can also be written as a direct sum

$$ds^2 = \Sigma_{\alpha=0}^3 \Sigma_{\beta=0}^3 dx(\alpha)\eta(\alpha,\beta)dx(\beta)$$

where $dx(0) = cdt$, $dx(1) = dx$ $dx(2) = dy$ and $dx(3) = dz$ and $\eta(\alpha, \beta) = \mathrm{diag}(1, -1, -1, -1)$

This hasn't really made life look any easier!!! but we're also going to use the Einstein summation convection: whereever an expression contains one index as a superscript and the SAME index as a subscript then the summation is implied. so we could write this a lot more neatly as $ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta$.

2.2 Notation

First lets start with standard vectors $\underline{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in standard cartesian coordinates. This notation is incredibly clumsy. So we're going to use instead a much more powerful and very compact notation. Axes are labeled by indices so $\{x, y, z\} = \{x^1, x^2, x^3\}$ where the superscripts DO NOT MEAN 'raised to the power' but instead pick out the components along the x, y and

z axes i.e. along the **i,j,k** directions which we are now going to call $\underline{e_1}, \underline{e_2}, \underline{e_3}$. If we need to raise to the power we will make it obvious e.g $(x^1)^3$. Then we can write this as

$$\underline{r} = x^1\underline{e_1} + x^2\underline{e_2} + x^3\underline{e_3}$$

It doesn't look much better yet. But we are going to get very sick of writing all the coordinates so we're going to use shorthand notation that $\{x^i\}$ means all of the $\{x^1, x^2, x^3\}$. We're also going to use the Einstein summation convection: whereever an expression contains one index as a superscript and the SAME index as a subscript then the summation is implied. So our vector

$$\underline{r} = x^1\underline{e_1} + x^2\underline{e_2} + x^3\underline{e_3} = x^i\underline{e_i} = \Sigma_{j=1}^3 x^j\underline{e_j} = x^j\underline{e}_j$$

It really doesn't matter what we call the index we sum over - if it occurs in the sum, we sum over it, whatever it is! But we'll have a convention that roman letters from the middle of the alphabet go from 1-3 (3D space, i,j...), while roman letters from the start of the alphabet (a,b,c...) mean $\{x^1, x^2..x^N\}$ - any N dimensional space and capital roman letters from the start of the alphabet (A,B....) mean 2D space. Greek letters are for spacetime, and go from 0-3, so $\{x^0, x^1, x^2, x^3\}$ where $x^0 = ct$, and $\{x^i\} = \{x, y, z\}$.

Once we have picked a given space(time) then we might do coordinate transformations within this. Up till now you've often seen this denoted as the primed coordinate frame, with the prime on the kernal letter, e.g. $x' = \gamma(x - vct/c)$ as the Lorentz transformation for the x coordinate I'm going to put a bar on the *index* to denote a transformation to a different frame (a prime can all to easily get mistaken for a 1, and I'm going to do it on the individual indices rather than on the kernal letter as we're going to generalise to things with more than 1 index...!!) so it looks like $x^{\bar{1}} = \gamma(x^1 - vx^0/c)$.