Symmetries of the Riemann Curvature Tensor

$$R^a{}_{bcd} = (\partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed})$$

We could write this out in full but its very long and tedious so instead we are going to use a trick. We can always define locally inertial coordinates, those in which the metric tensor is given by $\eta_{\alpha\beta}$ and the Christoffel symbols are zero, BUT THEIR DERIVATIVES ARE NOT - we can't transform gravity away in a global sense. I'd call this the primed frame but the equations will get very messy. BUT IN WHAT FOLLOWS WE ARE LOOKING AT THE COMPONENTS OF THE CURVATURE TENSOR IN THE LOCALLY INERTIAL FRAME.

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb}$$

Cyclic Identity

So you can see that so if we cyclically perturb the covariant indices then we have

$$R^a{}_{cdb} = \partial_d \Gamma^a{}_{bc} - \partial_b \Gamma^a{}_{dc}$$
$$R^a{}_{dbc} = \partial_b \Gamma^a{}_{cd} - \partial_c \Gamma^a{}_{bd}$$

so add all these, remembering that christoffel symbols are symmetric

$$R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \partial_d \Gamma^a{}_{bc} - \partial_b \Gamma^a{}_{dc} + \partial_b \Gamma^a{}_{cd} - \partial_c \Gamma^a{}_{bd} = 0$$

But $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$ is a TENSOR equation. So even though we derived it in the local frame, it must be true in ALL frames!

Index Symmetries

Start with the fully covariant quantity

$$R_{abcd} = g_{ae} R^e{}_{bcd}$$

$$R_{abcd} = g_{ae}[\partial_c \Gamma^e{}_{db} - \partial_d \Gamma^e{}_{cb}]$$

we can take the g_{ae} inside the derivatives and get

$$R_{abcd} = \partial_c \Gamma_{adb} - \partial_d \Gamma_{acb}$$

these Christoffel symbols can be written in terms of the derivatives of the metric (see lecture 7) where we had

$$2\Gamma_{abc} = \partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc}$$

so

$$2R_{abcd} = \partial_c[\partial_d g_{ab} + \partial_b g_{da} - \partial_a g_{db}] - \partial_d[\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{cb}]$$
$$R_{abcd} = \frac{1}{2}(\partial_c\partial_b g_{da} - \partial_c\partial_a g_{db} - \partial_d\partial_b g_{ca} + \partial_d\partial_a g_{cb})$$

where we've used the fact that $g_{ab} = g_{ba}$ and that $\partial_d\partial_c g_{ab} = \partial_c\partial_d g_{ab}$.

So then we can see that

$$R_{abcd} = -R_{bacd}$$

$$R_{abcd} = -R_{abdc}$$

$$R_{abcd} = R_{cdab}$$

Again, these are TENSOR equations, so must be true in all frames.

Bianchi Identity

Take the covariant derivative of the Riemann curvature tensor - but in a frame where the Christoffel symbols are zero then this is the same as the normal derivative!

$$R^a{}_{bcd;e} = \partial_e(\partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc}) = \partial_e\partial_c\Gamma^a{}_{bd} - \partial_e\partial_d\Gamma^a{}_{bc}$$

cyclically permuting c, d, e gives

$$R^a{}_{bde;c} = \partial_c\partial_d\Gamma^a{}_{be} - \partial_c\partial_e\Gamma^a{}_{bd}$$

$$R^a{}_{bec;d} = \partial_d\partial_e\Gamma^a{}_{bc} - \partial_d\partial_c\Gamma^a{}_{be}$$

So

$$R^a{}_{bcd;e}+R^a{}_{bde;c}+R^a{}_{bec;d} = \partial_e\partial_c\Gamma^a{}_{bd}-\partial_e\partial_d\Gamma^a{}_{bc}+\partial_c\partial_d\Gamma^a{}_{be}-\partial_c\partial_e\Gamma^a{}_{bd}+\partial_d\partial_e\Gamma^a{}_{bc}-\partial_d\partial_c\Gamma^a{}_{be} = 0$$

and again, its a tensor equation, so holds in all frames not just the locally inertial frame.