5 Schwarzschild metric

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = G^{\mu\nu} = \kappa T^{\mu\nu}$$

where

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$$

Einstein thought it would never be solved. His equation is a second order tensor equation - so represents 16 separate equations! Though the symmetry properties means there are 'only' 10 independent equations!!

But the way to solve it is not in full generality, but to pick a real physical situation we want to represent. The simplest is static curved spacetime round a spherically symmetric mass while the rest of spacetime is empty. Schwarzchild did this by guessing the form the metric should have

$$c^2d\tau^2 = A(r)c^2dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

so the $g_{\mu\nu}$ are not functions of t - field is static. And spherically symmetric as surfaces with r, t constant have $ds^2 = r^2(d\theta^2 + sin^2\theta d\phi^2)$.

Then we can form the Lagrangian and write down the Euler lagrange equations. Then by comparision with the geodesic equations we get the Christoffel symbols in terms of the unknown functions A and B and their radial derivatives $dA/dr = A'$ and $dB/dr = B'$. We can use these to form the Ricci tensor components as this is just defined from the christoffel symbols and their derivatives. And for EMPTY spacetime then $R_{\mu\nu} = 0$ NB just because the Ricci tensor is zero DOES NOT means that the Riemann curvature tensor components are zero (ie no curvature)!! Setting $R^{\nu\mu} = 0$ means that the equations are slightly easier to solve when recast into the alternative form

$$R^{\alpha\beta} = \kappa(T^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}T)$$

empty space means all the RHS is zero, so we do simply solve for $R^{\alpha\beta} = 0$. This gives $A = (1 + k/r)$ and $B = 1/A$.

Then we have to do some weak field limit connections to get the constant $k = 2GM/c^2$. Thus

$$ds^2 = c^2d\tau^2 = (1-2GM/c^2r)c^2dt^2-(1-2GM/c^2r)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2$$

or set $GM/c^2 = m$

$$ds^2 = c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

In this form we don't have to worry about κ but we can use the weak field approximation to connect it to gravity and get $\kappa = -8\pi G/c^4$.

The Schwarzschild metric is how gravity curves spacetime IFF the Einstein equations are correct! we saw that they are not derivable from first principles - they are merely the simplest way we can write gravity = curvature. So we need to see if this works, so we need to use this metric to figure out what this spacetime curvature (=gravity) predicts for paths of particles and photons which we can then MEASURE to test the theory.

5.1 Meaning of the Schwarzschild metric

We derived the metric from the Einstein equations, assuming that the cosmological constant is negligible and get

$$ds^2 = c^2 d\tau^2 = c^2(1 - 2m/r)dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

where $m = GM/c^2$. The $dt, dr, d\theta, d\phi$ are COORDINATES, not proper time, or proper distance. What do they mean ?

With $m = 0$ then the metric becomes simply Minkowski in spherical coordinates ie $ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$

Proper time is the time as measured by a clock which travels along with a particle - its obviously invariant as its a property of the particle NOT of the frame! Travelling WITH a particle means that there is no relative spatial motion so $c^2 d\tau^2 = c^2 dt^2$ ie $d\tau = dt$ ie proper time is coordinate time as measured by clocks which are stationary in the reference frame. Now turn m up, and $d\tau^2 = (1 - 2m/r)dt^2$. We are still going to say that clocks record proper time intervals along their world lines, and this clock is fixed, so we have that coordinate time is NOT equal to proper time $d\tau < dt$.

Proper distance is distance of an object measured at the same time so $dt = 0$. generally we supress the -ve sign in the metric and get proper distance as the spatial part of the metric so $dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$. For fixed θ and ϕ then $ds^2 = dr^2$ for $m = 0$. When we turn up m then the distances go strange - $dR^2 = (1 - 2m/r)^{-1} dr^2$. The proper distance ds IS NO LONGER given by dr, instead we have $dR = (1-2m/r)^{-1/2}dr > dr$. The way to do this is to curve space and can be visualised in an embeding diagram.

Asymptotically, both r and t go to proper distance and proper time.

But there is another feature of the metric which is immediately apparent:

$$ds^2 = c^2 d\tau^2 = c^2(1-2m/r)dt^2 - (1-2m/r)^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

which is that something very odd happens at $r = 2m$. the coefficient on radial coordinate $\rightarrow \infty$ while that for cooridate time $\rightarrow 0$. and worse happens for $r < 2m$, as the corefficients change sign. To understand this, lets see what happens to a stationary observer, held at fixed position by rockets. The the metric becomes $ds^2 = c^2 d\tau^2 = (1-2m/r)c^2 dt^2 < 0$. but the spacetime interval is then IMAGINARY - there are no real paths! The only real paths for $r < 2m$ MUST involve the spatial part of the metric changing fast enough to offset the -ve term which comes from coordinate time. So there are no such thing as stationary observers for $r < 2m$. Moving forward in time REQUIRES that you also move to smaller radius!! Everything goes down the hole.

5.2 Gravitational redshift

So we can instantly do gravitational redshift for stationary observers as these have proper time intervals which are given by $d\tau = (1-2m/r)^{1/2}dt$. So if we have light emitted at r_E and received at r_R then the time intervals each observer experiences are $d\tau_E = (1-2m/r_E)^{1/2}dt_E$ and $d\tau_R = (1-2m/r_R)^{1/2}dt_R$.

so how are coordinate time intervals related?

We send a light beam along a radial null geodesic. The beam is emitted at r_E and received at r_R. Then the path is

$$0 = c^2(1-2m/r)dt^2 - (1-2m/r)^{-1}dr^2$$

so it travels at coordinate speed

$$\frac{dr}{dt} = \pm c(1-2m/r)$$

so the coordinate time taken (use +ve if going outwards)

$$t_R - t_E = \frac{1}{c}\int_{r_E}^{r_R}(1-2m/r)^{-1}dr$$

we could evaluate this but we're not going to as the important this is that it only depends on space. If we send another signal a bit later from the same fixed position emitter to the same fixed position reciever then this coordinate time difference will be the same.

$$t_{R1} - t_{E1} = t_{R2} - t_{E2}$$

so then the coordinate time difference between the two emitted signals an the two received signals is also the same

$$t_{R1} - t_{R2} = t_{E1} - t_{E2} = \Delta t$$

But the clock at the point of emission records proper time not coordinate time. It is fixed in space, so the time it measures between the two signals is

$$\Delta\tau_E = (1 - 2m/r_E)^{1/2}\Delta t_E = (1 - 2m/r_E)^{1/2}\Delta t$$

but so is the clock at the receiver

$$\Delta\tau_R = (1 - 2m/r_R)^{1/2}\Delta t_R = (1 - 2m/r_R)^{1/2}\Delta t$$

so

$$\frac{\Delta\tau_E}{\Delta\tau_R} = \Big(\frac{1 - 2m/r_E}{1 - 2m/r_R}\Big)^{1/2}$$

Suppose the emitter were pulsating, then it has frequency $\nu \propto 1/\Delta\tau$. So

$$\frac{\nu_R}{\nu_E} = \Big(\frac{1 - 2m/r_E}{1 - 2m/r_R}\Big)^{1/2}$$

If $r_E < r_R$ then the photon crawls uphill so it loses energy so $\nu_R < \nu_E$. This can be measured - in the laboratory with a large tower as in the Pounds–Rebka–Snyder experiment. If light is unaffected by gravity then we can build an infinite energy machine where a particle dropped from a tower has rest mass plus mgh at the bottom, then converting mass to energy into a photon and send it back up the tower. if gravity doesn't affect light then it arives at the top with energy $h\nu = m_0c^2 + m_0gh$. And if we convert all this energy to mass then we get a particle of mass $m_1c^2 = m_0c^2 + m_0gh$ ie $m_1 > m_0$. Do this an infinite number of times and get infinite enrgy out!!! not a good plan. By contrast, gravitational redshift means that the photon loses the same amount of energy on the way up as the particle gained on the way down.

And gravitational redshift can be measured - either on earth by sending photons up a tower - though this is s TINY effect. Or look at absorption lines from the surface of a white dwarf and get a much bigger effect.

5.3 cyclic coordinates

To find geodesics paths (ie inertial frames!) then the first thing to do is write down the metric - and tailor it to the situation you want to solve. The do the Euler-Lagrange equations. Here we want general paths. These change in both ϕ and r but they are in a PLANE (θ=constant) so without loss of generality as the metric is spherically symmetric we can take this plane to be $\theta = \pi/2$ so the metric is then

$$ds^2 = c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2 d\phi^2$$

Geodesic paths so must satisfy the Euler-Lagrange equation where

$$L = 1/2 g_{\alpha\beta} \dot{x^\alpha} \dot{x}^\beta - (1 - 2m/r)c^2 \dot{t}^2 - (1 - 2m/r)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2$$

The E-L equations are

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$$

where dot denotes derivative with respect to proper time (which is an affine parameter). The easy equations will be those where a coordinate does not appear e.g. t and ϕ as then the Euler Lagrange equations only have one term

$$\frac{d}{d\tau} \Big(\frac{\partial L}{\partial \dot{x}^\alpha} \Big) = 0$$

so $\partial L / \partial \dot{x}^\alpha =$ constant. These are called cyclic coordinates.

Lets choose to do time coordinate $x^0 = ct$ first to see what its saying

$$\frac{\partial L}{\partial \dot{x}^0} = (1 - 2m/r)\dot{x}^0 = (1 - 2m/r)\frac{p^0}{\Delta}$$

since this is constant then if we find its value anywhere we know its value everywhere. At $r \to \infty$ this $\to p^0_\infty / \Delta = E_\infty / (\Delta c)$. so

$$(1 - 2m/r)\dot{x}^0 = (1 - 2m/r)c\dot{t} = E_\infty / (\Delta c)$$

$$(1 - 2m/r)c^2 \dot{t} = E_\infty / \Delta = E$$

i.e. E is energy per unit mass at infinity,