we are going to use the E-L equations in t rather than x^0, where

$$\frac{\partial L}{\partial \dot{t}} = (1-2m/r)c^2\dot{t} = E$$

from above

The other cyclic coordinate (ie one that doesn't appear in the metric) is ϕ. so the E-L equations here become

$$\frac{\partial L}{\partial \dot{x}^3} = \text{constant} = -r^2\dot{\phi}$$

This is indeed covariant angular momentum per unit mass - $p_\phi/\Delta = g_{\phi\phi}p^\phi/\Delta = -r^2\dot{\phi}$. More intuitively, this is (modulo a minus sign) like angular momentum $= \Delta r^2\omega = \Delta r^2\dot{\phi}$. Thus, angular momentum per unit mass $L_z = r^2\dot{\phi}$ But physically this all means is conservation of angular momentum!!!

What is this constant of motion?

$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)}{\partial \dot{x}^\alpha} = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial \dot{x}^\alpha}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}g_{\mu\nu}\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha}\dot{x}^\nu + \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha}$$

But the metric only depends on position not velocity ie only on x^α not on $\dot{x}^\alpha$ so $\partial g_{\mu\nu}/\partial \dot{x}^\alpha = 0$

$$= \frac{1}{2}g_{\mu\nu}\delta^\mu_\alpha\dot{x}^\nu + \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\delta^\nu_\alpha = g_{\mu\nu}\dot{x}^\mu = \dot{x}_\nu$$

But momentum is $p^\alpha = \Delta\dot{x}^\alpha$ where Δ is the particle mass. So this constant we are looking for is covariant momentum per unit mass. If coordinate does NOT appear in the metric, then that means that covariant momentum in that coordianate is conserved.

Lets do examples to make it all less abstract. Time coordinate $x^0 = ct$

$$\frac{\partial L}{\partial \dot{x}^0} = (1-2m/r)\dot{x}^0 = (1-2m/r)\frac{p^0}{\Delta}$$

since this is constant then if we find its value anywhere we know its value everywhere. At $r \to \infty$ this $\to p^0_\infty/\Delta = E_\infty/(\Delta c)$. so

$$(1-2m/r)\dot{x}^0 = (1-2m/r)c\dot{t} = E_\infty/(\Delta c)$$

$$(1-2m/r)c^2\dot{t} = E_\infty/\Delta = E$$

i.e. E is energy per unit mass at infinity, which is also equivalent to the zero component of covariant momentum = conservation of something like energy!!!

we are going to use the E-L equations in t rather than x^0, where

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5.4 Geodesic Paths - elliptical particle orbits

First thing to do is write down the metric - tailor it to the situation you want to solve. Here we want general paths. These change in both ϕ and r but they are in a PLANE (θ=constant) so without loss of generality as the metric is spherically symmetric we can take this plane to be $\theta = \pi/2$ so the metric is then

$$ds^2 = c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1}dr^2 - r^2 d\phi^2$$

Geodesic paths so must satisfy the Euler-Lagrange equations in the cyclic coordinates $(1 - 2m/r)c^2\dot{t} = E$ and $r^2\dot{\phi} = L_z$

So we have relations for $\dot{\phi}$ and $\dot{t}$. All we need now is one for $\dot{r}$ and we'd have everything we need (there are 3 coordinates so we need 3 equations!). We can get a relation for $\dot{r}$ from the Euler-Lagrance equations but its a bit nasty. All we really need is some way to relate $\dot{r}$ to the known $\dot{\phi}, \dot{t}$, and we can get such a relation more easily through the METRIC which for particles is:

$$c^2 = c^2(1 - 2m/r)\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$

rearrange to get

$$c^2(1-2m/r)\frac{\dot{t}^2}{\dot{\phi}^2} - (1-2m/r)^{-1}\Big(\frac{dr}{d\phi}\Big)^2 - r^2 = \frac{c^2}{\dot{\phi}^2}$$

substitute in from the EL equations for $\dot{t}, \dot{\phi}$,

$$\frac{c^2 r^4 E^2}{c^4(1-2m/r)L_z^2} - (1-2m/r)^{-1}\Big(\frac{dr}{d\phi}\Big)^2 - r^2 = \frac{c^2 r^4}{L_z^2}$$

multiply by $-(1-2m/r)$

$$\Big(\frac{dr}{d\phi}\Big)^2 = \frac{(E^2/c^4-1)c^2r^4}{L_z^2} + \frac{2mc^2r^3}{L_z^2} - r^2 + 2mr$$

compare this to Newtonian. total energy (KE+PE) per unit mass is $\tilde{E} = \frac{1}{2}v^2 - GM/r$. in spherical polars then $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$ so

$$\tilde{E} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - GM/r$$

again $r^2\dot{\phi} = L_z$ constant ang. momentum and $GM/c^2 = m$ so

$$\frac{2\tilde{E}}{\dot{\phi}^2} = \Big(\frac{dr}{d\phi}\Big)^2 + r^2 - \frac{2mc^2}{r\dot{\phi}^2}$$

$$\frac{2\tilde{E}r^4}{L_z^2} = \Big(\frac{dr}{d\phi}\Big)^2 + r^2 - \frac{2mc^2r^3}{L_z^2}$$

$$\Big(\frac{dr}{d\phi}\Big)^2 = \frac{2\tilde{E}r^4}{L_z^2} + \frac{2mc^2r^3}{L_z^2} - r^2$$

so this is the SAME (as long as energy means something slightly different $2\tilde{E} = (E^2/c^4-1)c^2$ - but we'd kind of expect as Newtonian energy doesn't include rest mass and isn't relativistic) EXCEPT for the $2mr$ term in the GR equation which is NOT PRESENT in the Newtonian! so orbits in GR should be just a little bit different - this is a TEST of GR, of Einsteins equations which are the simplest way to write gravity=curvature.

5.5 Precession of perihelion of Mercury

need to solve these horrid equations. nowadays we can simply put them into maple/mathematica and/or write a computer program for them. but it can be done analytically by hacking through lots of tediuos algebra. what happens is that for small perturbations from circular orbits. These small deviations from circularity $y = u - u0$ where $u = 1/r$ and $u0 = 1/r0$ where $r0$ is the mean distance. $y = A\sin(\phi - \phi_0)$ so these deviations which make the orbit an ellipse are periodic on 2π. The orbit comes back to the same r after 2π so its CLOSED.

But the GR equation has an extra term (and a different definition of energy). If we do the same thing then neglect terms in y^3 - these are SMALL deviations then $y = y_0 + A\cos(k\phi - \phi_0)$ with $k = (1 - \frac{6G^2M^2}{L_z^2c^2})^{1/2} \approx 2\pi(1 + \frac{3G^2M^2}{L_z^2c^2})$ from a binomial expansion. So we get round to the same r NOT when $\phi = 2\pi$ but when $k\phi = 2\pi$. The orbit is NOT closed. the perihelion precesses.

So first we need this constant $L_z = r^2\dot{\phi} = r^2 2\pi/P$ where P is the period. In Newtonian this is $P^2 = 2\pi r^3/(GM)$ so $L_z^2 = r^4 GM/r^3 = GMr$ so $\Delta\phi = 6\pi G^2M^2/(GMrc^2) = 6\pi GM/(rc^2)$. For Mercury (closest planet so one in the strongest gravitational field, ie where this effect will be biggest) then $r \sim 5.2 \times 10^{10} m$ so $\Delta\phi = 5 \times 10^{-7}$ radians per orbit. This is TINY!! but its also cumulative. One orbits precession is difficult to measure, but after 100 years ie 415 orbits then the point of closest approach (perihelion) has shifted by 0.01189 radians or 43 arcseconds. This is measurable, and was a well known problem in the 19th century that newtonian mechanics didn't get the right answer for Mercury's orbit. GR does!!!

5.6 Geodesic paths - photon orbits

We can do these in an identical manner, with the E-L equations in the cyclic coordinates as $(1 - 2m/r)\dot{t} = E/c^2$ and $r^2\dot{\phi} = L_z$. the only difference now is the metric - photons travel on NULL geodesics so proper time is zero, and dot denotes derivative with respect to some affine parameter.

$$c^2(1 - 2m/r)\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0$$

substitute in from the EL equations for $\dot{t}, \dot{\phi}$,

$$\frac{r^4 c^2 E^2}{(1-2m/r)c^4 L_z^2} - (1-2m/r)^{-1}\Big(\frac{dr}{d\phi}\Big)^2 - r^2 = 0$$

$$\Big(\frac{dr}{d\phi}\Big)^2 + r^2(1-2m/r) - \frac{E^2 r^4}{c^2 L_z^2} = 0$$

$$\frac{1}{u^4}\Big(\frac{du}{d\phi}\Big)^2 + \frac{1}{u^2}(1-2mu) - \frac{E^2}{c^2 L_z^2 u^4} = 0$$

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = \frac{E^2}{c^2 L_z^2} + 2mu^3$$

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = \frac{E^2}{c^2 L_z^2} + \frac{2GMu^3}{c^2}$$

The Newtonian prediction is actually a little ambiguous. Newtonian gravity affects things with mass, so NOT photons, but acceleration is MASS INDEPENDANT, so maybe SHOULD affect photons. But if we can accelerate photons then we can change their velocity, yet special relativity says we can't accelerate photons, we can't go faster than c. So by the time people were thinking about this, they were pretty much convinced that gravity didn't affect photons so light should travel in a straight line. and we can get flat space by setting $m = GM/c^2 \to 0$ so

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = u_0^2$$

where $u_0^2 = E^2/(c^2 L_z^2)$. The equation for this in terms of u and ϕ would be $u = u_0 \sin\phi$ where u_0 is the value of u at closest approach, $r_0 = 1/u_0$ which is also called an impact parameter.

The full GR one has an extra term $2mu^3$. This is very small, so call it ϵu^3 just to make this explicit. we can expand and what comes out is

$$u = u_0(1 - \frac{1}{2}\epsilon u_0)\sin\phi + \frac{1}{2}\epsilon u_0^2(1-\cos\phi)^2$$

so we no longer expect a straight line as the photon goes from $\phi = 0$ to π. As $r \to \infty$ i.e. $u \to 0$ then $\phi \to \pi + \alpha$ where $\alpha \ll 1$. $\cos(\pi+\alpha) = \cos\pi\cos\alpha - \sin\pi\sin\alpha \approx -1$ while $\sin(\pi+\alpha) = \sin\pi\cos\alpha + \cos\pi\sin\alpha = -\sin\alpha \approx -\alpha$.

$$0 = -(1 - \frac{1}{2}\epsilon u_0)\alpha + 2\epsilon u_0 \approx -\alpha + 2\epsilon u_0$$

$$\alpha \sim 2\epsilon u_0 = 4\frac{GM}{r_0 c^2}$$

Take r_0 as equal to the suns radius and look at a star in the line of sight just away from the limb of the sun and then the prediction for the deflection of its position is 1.75 seconds of arc. This was the first test of GR, measuring positions of stars close to the sun as seen during a solar eclipse and comparing these to their positions as seen when the sun was nowhere near the line of sight.