5.12 Nature of the Event Horizon: radial geodesics for particles

$$ds^2 = c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2$$

our standard E-L equations are $(1-2m/r)c^2\dot{t} = E$ where E is the energy per unit mass measured at infinity. this is a constant, so if we measure it anywhere we have measured it everywhere. drop from rest at $r \to \infty$ so in the metric, $dr = 0$ is drop from rest, so $c^2 d\tau^2 = (1 - 2m/\infty)c^2 dt^2 2$ and $\dot{t} = 1$ (proper time is the same as coordinate time for someone stationary at infinity). so then the E-L equation is $(1 - 2m/r)c^2\dot{t} = E$ so $c^2 = E$ (energy at infinity for dropping from stationary is just rest mass!)

so now we follow it down:

$$c^2 = (1 - 2m/r)c^2\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2$$

what we want to do is compare the speed of the satellite as measured by someone at infinity, to the speed of the satellite as measured by someone in the satellite.

someone in the satellite measures $dr/d\tau = \dot{r}$ for the rate at which they change in r going past stationary markers.

$$c^2(1 - 2m/r) = (1 - 2m/r)^2 c^2 E^2/[c^4(1 - 2m/r)^2] - \dot{r}^2$$

$$\dot{r}^2 = c^4/[c^2 - (1 - 2m/r)c^2 = 2mc^2/r$$

hence $\dot{r} = \pm\sqrt{2m/r}c$ where we take the -ve sign as we are moving inwards

What about someone at infinity? We know that a stationary observer at infinity measures $d\tau = dt$ so the speed they measure going past the markers in r is dr/dt.

$$c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2$$

$$c^2/\dot{t}^2 = (1 - 2m/r)c^2 - (1 - 2m/r)^{-1}(dr/dt)^2$$

the E-L equations gives us $\dot{t} = 1/(1-2m/r)$ as $E = c^2$ for a satellite dropped from rest so

$$c^2(1 - 2m/r)^2 = (1 - 2m/r)c^2 - (1 - 2m/r)^{-1}(dr/dt)^2$$

$$c^2(1-2m/r)^3 = (1-2m/r)^2c^2 - (dr/dt)^2$$

$$(dr/dt)^2 = c^2(1-2m/r)^3 - (1-2m/r)^2c^2 = (1-2m/r)^2(2m/r)c^2$$

so for someone in the satelite, they keep on accelerating, reaching c at the horizon, and keep on accelerating beyond the horizon! whereas as seen from infinity they reach maximum velocity at $r = 6m$ and then SLOW DOWN and eventually stop at the horizon.

so we can ask how long it takes someone on the spacecraft to fall from $r = 6m$ to $r0$ and this is

$-\int_{6m}^{r0}\sqrt{r/2m}dr = \int_0^T cd\tau$

life gets easier if we go to dimensionless radii $x = r/m$ so $mdx = dr$

so $-\sqrt{1/2}[x^{3/2}/(3/2)]_6^{x0} = cT/m$ where again cT/m is time measured in dimensionles units (speed=distance/time, and m is a distance - its half the size of the event horizon)

and hence $\sqrt{2}[(6)^{3/2} - x0^{3/2}]/3 = cT/m$.

Thus this is a perfectly finite time to drop to $r0 = 2m$ and equally finite to drop from $r = 6m$ to $r = 0$ -its $cT/m = 6\sqrt{6} = 6.93$

but the coordinate time to drop this same change in radius is

$$\int_{6m}^{r0} -\frac{1}{\sqrt{2m}}\frac{r^{1/2}}{1-2m/r}dr = cT$$

again set $x = r/m$ so $dx = dr/m$

$$-m/\sqrt{2}[1/3(\sqrt{(}2x)(x+6) - 12\tanh^{-1}(\sqrt{(}x/2))]_{6m}^{r0} = cT$$

$$1/3(\sqrt{(}12)(12) - 12\tanh^{-1}(\sqrt{(}3)) - \sqrt{(}2x0)(x0+6) + 12\tanh^{-1}(\sqrt{(}x0/2))) = cT/m$$

so we are into special functions, but they hold no terrors for us! $\tanh^{-1} z = 1/2\log(1+z)/(1-z)$.

$$= 1/3(\sqrt{(}12)(12) - \sqrt{(}2x0)(x0+6) - 6\log(1+\sqrt{3})/(1-\sqrt{3}) + 6\log(1+\sqrt{x0/2})/(1-\sqrt{x0/2})$$

$$= 1/3(\sqrt(12)(12) - \sqrt(2x0)(x0+6) + 6\log[(1-\sqrt{3})(1+\sqrt{x0/2})]/[(1+\sqrt{3})(1-\sqrt{x0/2})]$$

interesting!! we have logs of -ve numbers here...but there are two so its positive definite if we switch both around - IFF $x0/2 > 1$

$$= 1/3(\sqrt(12)(12) - \sqrt(2x0)(x0+6) + 6\log[(\sqrt{3}-1)(1+\sqrt{x0/2})]/[(1+\sqrt{3})(\sqrt{x0/2}-1)]$$

As $x0 \to 2$ ie as we approach the horizon, the denominator goes to zero, and we have $\log\infty \to \infty$ so it takes an infinite amount of time to fall to the event horizon, and never reaches the singularity!!

in fact it takes even longer than this as we have to add in the light travel time, which itself $\to \infty$ as $r \to 2m$. the time we see at infinity for it to fall from $6m$ to $r0$ is the coordinate time it takes the satellite to fall, plus the light travel time for it to come from r0 to $6m$. This is

we know the coordinate speed for light is $dr/dt = \pm(1-2m/r)c$ so now we need the +ve sign as the light goes outwards. so this is

$$\int_{r0}^{6m} dr/(1-2m/r) = 6 - x0 + 2\log(6-2)/(x0-2) = cT_{light}/m$$

$$6 - x0 + 2\log 4/(x0-2) = cT_{light}$$

so we are seeing the same thing happen - the denominator of the log goes to 0 so this also goes as $\log\infty \to \infty$ as $x0 \to 2$

we calculated the grid time for the satellite to drop from $6m$ to $r0$ plus the grid time for light to travel back from $r0$ to $6m$. but both these $\to \infty$ as $r2 \to 2m$. so it takes an infinite amount of coordinate time to get out. But coordinate time IS time as measured by a distant observer, so this means that it takes an infinite amount of time for the distant observer to recieve light emitted from close to the event horizon. so an observer at infinity never sees the satellite even get to the event horizon in a finite time, let alone the singularity!

But someone INSIDE the satellite measures a finite time to reach the horizon and a finite time to fall from there to the the singularity. These are very different pictures of what must be a single reality!!!

5.13 Nature of the Event Horizon: acceleration

worse happens when we try instead to look at acceleration as measured by stationary observers. we can probe acceleration by looking at dropping the spacecraft from rest at $r0$ rather than $r \to \infty$ and then looking at the velocity as measured by a stationary observer at $r < r0$.

if we drop from rest at $r0$ rather than $r \to \infty$ then our metric at $r0$ is $c^2 = (1 - 2m/r0)c^2\dot{t}^2$ since we are at rest so there is no $dr, d\phi$. and then $\dot{t} = \pm(1 - 2m/r0)^{-1/2}$ and so the constant E in the euler lagrange equation at $r0$ is

$$(1 - 2m/r0)(1 - 2m/r0)^{-1/2} = E/c^2$$

$(1 - 2m/r0)^{1/2} = E/c^2$ hence $(1 - 2m/r)\dot{t} = (1 - 2m/r0)^{1/2}$