2.2 The Event Horizon (cont)

Lets pull it all together. At $r = 2m$ there is infinite acceleration, so no such thing as fixed anything. So its NOT just a coordinate singularity, physically there is something going on here, and this is supported by an observer at infinity watching an infalling particle - they see it initially accelerate towards the hole but as it gets closer then it slows down and appears to STOP at the horizon. But its not a true singularity - on an infalling particle it takes a finite time to get to $r = 0$ rather than running into any real barrier at $r = 2m$.

The way to think about this is to look at the scalar we can form from the Riemman curvature tensor $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$. This is NOT the same as the scalar we formed from contracting the Ricci tensor $R^{\mu}{}_{mu} = g_{\mu\nu}R^{\mu\nu}$ - We saw that for empty space outside a gravitating body then this (and the Ricci tensor itself) were ZERO as the stress energy tensor and scalar are zero if space is EMPTY and the Einstein equations can be written $R^{\mu\nu} = \kappa(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T)$. But the ricci tensor is a contraction of the full Riemann curvature and it is NOT necessarily true that all components of the Riemann curvature are zero if $R^{\mu\nu} = 0$ - there is real curvature of spacetime (there has to be as this is GRAVITY). This Riemann curvature scalar $\propto r^{-6}$. There IS a true curvature singularity, and its at $r = 0$ as expected.

If we draw the curvature of spacetime (rather than an embedding diagram which shows proper distance versus r) then we can see whats going on. Any freefalling observer is in an inertial frame - we can draw a flat bit of spacetime which is tangent to the spacetime curvature at that point. and since time is orthoganal to space then we can use this to define the t axis. and lightcones make a 45 degree angle as normal - its all locally intertial and fine and special relativity works, but its TILTED with respect to ∞ due to the spacetime curvature. at $r = 2m$ the tilt is 45 degrees, so light DOES NOT GET OUT. There is no way that lightcones can get out to infinity - they simply don't connect. so a black hole is simply spacetime so curved that not even light can get out.

2.3 Appearance of a black hole from ∞

lets just play a bit with light round a black hole - how close can we get with lightbeams before they get inexorably dragged down the hole?

back to the schwarzschild metric in the equatorial plane - do the lagrangian and euler lagrange equations to find the geodesic paths (=natural paths)

$$ds^2 = c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2 d\phi^2$$

$$L = \frac{1}{2}[(1 - 2m/r)c^2 \dot{t}^2 - (1 - 2m/r)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2]$$

cyclic coordinate t: $\partial L/\partial t = 0$ so $\partial L/\partial \dot{t} = \text{constant} = (1 - 2m/r)c^2 \dot{t}$. we can absorb the c^2 in the constant term and write $(1 - 2m/r)\dot{t} = E$. Similarly with the other cyclic coordinate ϕ we have $\partial L/\partial \dot{\phi} = \text{constant} = -r^2 \dot{\phi}$. We can absorb the -ve sign in the constant and write $r^2 \dot{\phi} = cL_z$ where we've put a c in our constant because then it'll look a lot nicer at the end. substitute these back in the metric and we have for LIGHT (null geodesics)

$$0 = \frac{(1 - 2m/r)c^2 E^2}{(1 - 2m/r)^2} - (1 - 2m/r)^{-1} \dot{r}^2 - \frac{r^2 c^2 L_z^2}{r^4}$$

$$0 = c^2 E^2 - \dot{r}^2 - (1 - 2m/r)c^2 L_z^2 / r^2$$

$$\dot{r}^2 = \frac{c^2}{L_z^2}\Big[\frac{E^2}{L_z^2} - \frac{(1 - 2m/r)}{r^2}\Big] = \frac{c^2}{L_z^2}\Big(\frac{E^2}{L_z^2} - V^2(r)\Big)$$

where $V^2(r) = (1 - 2m/r)/r^2$. Thus the photon can only get everywhere if $\dot{r}$ exists for all r. This is the case if $E^2/L_z^2 > V^2(r)$. But $V^2(r)$ has a maximum - differentiate and get

$$dV^2(r)/dr = (-2/r^3)(1 - 2m/r) + 2m/r^2 = 0$$

i.e. at $r = 3m$. at this point $V^2(r) = (1 - 2m/3m)/(9m^2) = 1/(27m^2)$ if $E^2/L_z^2 > V^2(3m)$ then $\dot{r}$ exists for all r i.e. the photon can get to all r - the angular momentum barrier is not big enough to stop it falling down the black hole.

But E^2/L_z^2 has a nice physical meaning - for light we saw that

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = \frac{E^2}{L_z^2} + 2mu^3$$

where $u = 1/r$. The solution for this for $m \to 0$ is $u = u_0 \sin\phi$ i.e. a straight line with distance of closest approach (better name is IMPACT parameter)

$r_0 = 1/u_0 = L_z/E$. so our black hole we would see at infinity that light which set out with inpact parameter $r^2 < 27m^2$ ie $r < \sqrt{27}m$ would go down the black hole. This is BIGGER than you'd expect for the size of the photon orbit $r = 3m$ and the event horizon at $r = 2m$. The black hole dark spot looks bigger because it bends light!