3.4 notes on notation!

partial derivative $\partial A^a/\partial x^c = \partial_c A^a = A^a{}_{,c}$ (I won't use the latter as its too easy to lose a comma!)
covariant derivative $\nabla_c A^a = A^a{}_{;c}$ (I will use the semicolon notation because physicists tend to freeze when they see del!!)
absolute derivative $\nabla_{\underline{s}} A^a = DA^a/ds$
Christoffel symbols (or connection coefficients) $\Gamma^a{}_{bc}$ or $\left\{ {a \atop bc} \right\}$ or $\{a, bc\}$

3.5 Example: 2D flat space

The metric for flat space in cartesian coordinates $g_{AB} = \text{diag}(1,1)$ DOES NOT DEPEND ON POSITION. So the partial derivatives of the metric are ZERO. so $\Gamma^A{}_{BC} = 0$.

BUT this is not true if we'd done it in terms of polars, $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$ where $g_{AB} = \text{diag}(1, r^2)$. Here the derivaties are NOT zero.

So Christoffel symbols are like the metric - they do tell us about curvature (what we are interested in) but they also tell us about what coordinate system we have chosen (which isn't at all fundamental). so if we are going to bother calculating them, we may as well choose a real curved surface to play with.

3.6 Surface of a sphere

before we worked out the metric components for flat space

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

we can describe a sphere just by saying we are at fixed radius $r = a$ so $dr = 0$. so

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2$$

i.e. $g_{AB} = \text{diag}(a^2, a^2\sin^2\theta)$

Christoffel symbols are defined as

$$\Gamma^f_{ca} = \frac{1}{2} g^{fb}(\partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca})$$

so we need to also find the covariant metric components g^{AB} from $g^{AB} g_{BC} = \delta^A_C$. In general this leads to a set of simultaneous equations to solve

$$g^{AB} g_{BC} = g^{A1} g_{1C} + g^{A2} g_{2C} = \delta^A_C$$

BUT for diagonal metrics we don't have to do a matrix inverse to solve these! in a diagonal metric all cross terms are zero so we get $g^{AB}g_{BA} = g^{AA}g_{AA} = 1$ or $g^{AA} = 1/g_{AA}$ ONLY FOR DIAGONAL METRICS!

$$g^{AB} = \text{diag}[1/a^2, 1/(a^2 \sin^2\theta)]$$

So in our 2D space we have

$$\Gamma^\theta_{\theta\theta} = 1/2g^{\theta B}(\partial_\theta g_{\theta B} + \partial_\theta g_{B\theta} - \partial_B g_{\theta\theta})$$

as $g^{\theta B} = 0$ except for $B = \theta$ then this collapses to

$$\Gamma^\theta_{\theta\theta} = 1/2g^{\theta\theta}(\partial_\theta g_{\theta\theta} + \partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta}) = 1/2\ .\ 1/a^2\ .\ 0 = 0$$

$$\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = 1/2g^{\theta B}(\partial_\phi g_{\theta B} + \partial_\theta g_{B\phi} - \partial_B g_{\theta\phi})$$

again $g^{\theta B}$ is only nonzero for $B = \theta$ since the metric is diagonal so

$$\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\theta\phi} = 1/2g^{\theta\theta}(\partial_\phi g_{\theta\theta} + \partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\phi}) = 1/2\ .\ 1/a^2\ .\ 0 = 0$$

$$\Gamma^\theta_{\phi\phi} = 1/2g^{\theta B}(\partial_\phi g_{\phi B} + \partial_\phi g_{B\phi} - \partial_B g_{\phi\phi}) = 1/2\ .\ 1/a^2(0 + 0 - \partial_\theta g_{\phi\phi})$$

$$= 1/2\ .\ 1/a^2(-2a^2 \sin\theta\cos\theta) = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\theta\theta} = 1/2g^{\phi B}(\partial_\theta g_{\theta B} + \partial_\theta g_{B\theta} - \partial_B g_{\theta\theta}) = 1/2g^{\phi\phi}(\partial_\theta g_{\theta\phi} + \partial_\theta g_{\theta\phi} - \partial_\phi g_{\theta\theta}) = 0$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = 1/2g^{\phi B}(\partial_\phi g_{\theta B} + \partial_\theta g_{B\phi} - \partial_B g_{\theta\phi}) = 1/2g^{\phi\phi}(\partial_\phi g_{\theta\phi} + \partial_\theta g_{\phi\phi} - \partial_\phi g_{\theta\phi})$$

$$= 1/2\ 1/(a^2 \sin^2\theta)\partial_\theta g_{\phi\phi} = 1/2\ 1/(a^2 \sin^2\theta) 2a^2 \sin\theta\cos\theta = \cot\theta$$

$$\Gamma^\phi_{\phi\phi} = 1/2g^{\phi B}(\partial_\phi g_{\phi\phi} + \partial_\phi g_{\phi\phi} - \partial_\phi g_{\phi\phi}) = 0$$

3.7 Geodesic equations

And now we have everything we need to get the geodesic paths (inertial frames). If we were differential geometers, then we would define a geodesic as the shortest distance between two points. But we are phycisics so we can use the alternative definition that these are inertial frames. and in an inertial frame the velocity DOES NOT CHANGE - there are no forces to prodsce an acceleration. So velocity $\underline{v} = v^\alpha \underline{e}_\alpha = dx^\alpha/d\tau \underline{e}_\alpha$ If there is no change in this then its derivative is zero but we also saw that there can be swings in a vector which arise from curved space. In other words we have to cast this in tensor language and say its the ABSOLUTE derivative which is zero in order to simply parallel transport the velocity vector. So the components can change like

$$\frac{dv^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$
$$\frac{d}{d\tau}\frac{dx^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$
$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

or using notation where dot means derivative w.r.t. τ

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$$

This works for any parameter u linearly related to path length s i.e. for $u = A + Bs$. Then the **affinely parameterized geodesic equation** in an N dimensional manifold is

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0$$

or in slighly cleaner notation, where dot denotes derivative w.r.t. s

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$$

So now we know how to do geodesic paths, as well as derivatives.

3.8 Example: Geodesics in flat space

we saw that in flat space all the $\Gamma^i{}_{jk} = 0$ so the geodesic equations becomes

$$\ddot{x}^a = 0$$

were dot denotes derivative wrt path length s. integrate to get $\dot{x}^a = A$ and $x^a = As + B$ so geodsics in flat space have constant velocity and direction - Newtonian inertial frame (constant motion, straight line!)

3.9 Example: Geodesics on a sphere

in general its too hard to do geodesics in full generality. so often we just choose a path and see if its a geodesic! and a natural one to choose is a path defined by only one of the parameters - its called a parameter curve.

so on our sphere, we know that for a sphere the only non-zero christoffel symbols are $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$, and $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$.

Geodesic paths satisfy the equation

$$\frac{dx^A}{ds^2} + \Gamma^A_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = 0$$

suppose the path s is just a change in θ then $s = a\theta$ so there is no dependance on ϕ i.e. $x^2 = \phi$constant so $d\phi/ds = 0$. While for θ we have $d\theta/ds = 1/a$ and $d^2\theta/ds^2 = 0$. For θ then

$$\frac{d^2\theta}{ds^2} + \Gamma^{\theta}_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = \frac{d^2\theta}{ds^2} + \Gamma^{\theta}_{\phi\phi}\frac{d\phi}{ds}\frac{d\phi}{ds} = 0$$

so this looks good. Check also that our condition for ϕ holds

$$\frac{d^2\phi}{ds^2} + \Gamma^{\phi}_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = \frac{d^2\phi}{ds^2} + \Gamma^{\phi}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} + \Gamma^{\phi}_{\phi\theta}\frac{d\phi}{ds}\frac{d\theta}{ds} = 0$$

so all such paths are geodesics. - these are GREAT CIRCLES.