3.1 Summary: Tensor derivatives

Absolute derivative of a contravariant tensor over some path

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \lambda^b \Gamma^a_{bc} \frac{dx^c}{ds}$$

gives a tensor field of same type (contravariant first order) in this case. The first bit takes into account both real physical changes in λ^a AND the way the curvature of the space can swing the vector while the second one takes out the swinging the vector from the curvature of space alone. So if there is no physical change then $D\lambda^a/ds = 0$ and the vector is simply parallelly transported. Absolute derivative obeys all the normal rules for derivatives so $D(\lambda^a + k\mu^a)/ds = D(\lambda^a)/ds + kD(\mu^a)/ds$ (linear) and $D(\lambda^a\mu^a)/ds = \mu^a D(\lambda^a)/ds + \lambda^a D(\mu^a)/ds$ (Leibniz' rule).

Absolute derivative of a scalar

There is no space swing with a simple number, ϕ. so $D\phi/ds = d\phi/ds$

Absolute derivative of covariant tensors

$D\phi/ds = d\phi/ds$ so let $\phi = \lambda^a \mu_a$ then

$$\frac{D\phi}{ds} = \frac{d\phi}{ds} = \frac{D(\lambda^a\mu_a)}{ds} = \mu_a \frac{D\lambda^a}{ds} + \lambda^a \frac{D\mu_a}{ds}$$

$$\frac{d(\lambda^a\mu_a)}{ds} = \mu_a \Big(\frac{d\lambda^a}{ds} + \lambda^b \Gamma^a_{bc} \frac{dx^c}{ds}\Big) + \lambda^a \frac{D\mu_a}{ds}$$

$$\lambda^a \frac{d\mu_a}{ds} + \mu_a \frac{d\lambda^a}{ds} = \mu_a \frac{d\lambda^a}{ds} + \mu_a \lambda^b \Gamma^a_{bc} \frac{dx^c}{ds} + \lambda^a \frac{D\mu_a}{ds}$$

$$\lambda^a \frac{D\mu_a}{ds} = \lambda^a \frac{d\mu_a}{ds} - \mu_a \lambda^b \Gamma^a_{bc} \frac{dx^c}{ds}$$

do some index manipulation and get

$$\frac{D\mu_a}{ds} = \frac{d\mu_a}{ds} - \Gamma^b_{ac} \mu_b \frac{dx^c}{ds}$$

Absolute derivative of higher order tensors

For example, to get the absolute derivative of a mixed tensor τ^a_b then look at the special case where $\tau^a_b = \lambda^a \mu_b$

$$\frac{D\tau^a_b}{ds} = \lambda^a \frac{D\mu_b}{ds} + \mu_b \frac{D\lambda^a}{ds} = \lambda^a \Big(\frac{d\mu_b}{ds} - \Gamma^d_{bc} \mu_d \frac{dx^c}{ds}\Big) + \mu_b \Big(\frac{d\lambda^a}{ds} + \Gamma^a_{dc} \lambda^d \frac{dx^c}{ds}\Big)$$

$$= \frac{d(\lambda^a \mu_b)}{ds} + \mu_b \Gamma^a_{dc} \lambda^d \frac{dx^c}{ds} - \lambda^a \Gamma^d_{bc} \mu_d \frac{dx^c}{ds} = \frac{d(\tau^a_b)}{ds} + \Gamma^a_{dc} \tau^d_b \frac{dx^c}{ds} - \Gamma^d_{bc} \tau^a_d \frac{dx^c}{ds}$$

Covariant derivative

$$\lambda^a_{;c} = \frac{\partial \lambda^a}{\partial x^c} + \Gamma^a_{bc} \lambda^b$$

rules are as above, so for a second order contravariant tensor we have

$$\tau^{ab}{}_{;c} = (\lambda^a \mu^b)_{;c} = \lambda^a{}_{;c} \mu_b + \lambda^a \mu_{b;c}$$

$$= \frac{\partial \lambda^a}{\partial x^c} \mu^b + \Gamma^a_{dc} \lambda^d \mu^b + \lambda^a \frac{\partial \mu^b}{\partial x^c} + \lambda^a \Gamma^b_{dc} \mu^d = \partial_c \tau^{ab} + \Gamma^a_{dc} \tau^{db} + \Gamma^b_{dc} \tau^{ad}$$

to find covarient derivative, again we use the fact that a scalar has no space swing so the covariant derivative is the same as partial derivative $\phi_{;c} = \partial_c \phi /$, while we can write $\phi = \lambda^a \mu_a$ so
$\phi_{;c} = \partial_c \phi / = (\lambda^a \mu_a)_{;c} = (\lambda^a{}_{;c}) \mu_a + \lambda^a (\mu_{a;c})$
$\partial_c \phi = (\partial_c \lambda^a) \mu_a + \lambda^a (\partial_c \mu_a) = (\partial_c \lambda^a + \Gamma^a{}_{dc} \lambda^d) \mu_a + \lambda^a \mu_{a;c}$
$\lambda^a \partial_c \mu_a = \Gamma^a{}_{dc} \lambda^d \mu_a + \lambda^a \mu_{a;c}$
relabel indices to get all λ components as λ^a so $\lambda^a \partial_c \mu_a = \Gamma^b{}_{ac} \lambda^a \mu_b + \lambda^a \mu_{a;c}$
$\mu_{a;c} = \partial_c \mu_a - \Gamma^b{}_{ac} \mu_b$

we can can do higher order covariant derivatives similarly

$$\lambda_{ab;c} = \frac{\partial \lambda_{ab}}{\partial x^c} - \Gamma^d_{ac} \lambda_{db} - \Gamma^d_{bc} \lambda_{ad}$$

Covariant differention forms a tensor field of 1 higher covariant order than the original tensor field.

Covariant derivative of the metric

In getting the Christoffel symbols (section 3.4) in terms of the metric we had

$$\frac{\partial g_{ab}}{\partial x^c} = \frac{\partial \vec{e_a} . \vec{e_b}}{\partial x^c} = \frac{\partial \vec{e_a}}{\partial x^c} . \vec{e_b} + \vec{e_a} . \frac{\partial \vec{e_b}}{\partial x^c}$$
$$= \Gamma^d_{ac} \vec{e_d} . \vec{e_b} + \vec{e_a} . \vec{e_d} \Gamma^d_{bc} = \Gamma^d_{ac} g_{db} + \Gamma^d_{bc} g_{ad}$$

rearrange this to get

$$\frac{\partial g_{ab}}{\partial x^c} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} = 0$$

but this is the definition of the covariant derivative of a second order covarient tensor. so $g_{ab;c} = 0$, and we can similarly get that $g^{ab}_{;c} = 0$. This means that index lowering can be swapped in and out of covariant differention. $R_a = g_{ab}R^b$. Then $R_{a;c} = (g_{ab}R^b)_{;c} = g_{ab;c}R^b + g_{ab}R^b{}_{;c} = g_{ab}R^b{}_{;c}$

likewise $g^{ab}{}_{;c} = 0$ so index raising can also be swapped in and out of covariant differentiation