3.14 Summary: Tensor derivatives (cont)

Covariant derivative

$$\lambda^a{}_{;c} = \frac{\partial \lambda^a}{\partial x^c} + \Gamma^a_{bc}\lambda^b$$

Covariant differention forms a tensor field of 1 higher covariant order than the original tensor field. Rules are the same as for absolute derivative.

so for a scalar $\phi = \lambda^a \mu_a$, then $\phi_{;c} = \partial_c \phi$ (ie no spatial terms as a number is just a number) so

$$\phi_{;c} = \partial_c \phi = (\lambda^a \mu_a)_{;c} = \lambda^a \mu_{a;c} + \mu_a \lambda^a{}_{;c}$$

$$\lambda^a \partial_c \mu_a + \mu_a \partial_c \lambda^a = \lambda^a \mu_{a;c} + \mu_a(\partial_c \lambda^a + \Gamma^a{}_{bc}\lambda^b)$$

$$\lambda^a \partial_c \mu_a = \lambda^a \mu_{a;c} + \mu_a \Gamma^a{}_{bc}\lambda^b$$

do index switching on the last term to get it all onto λ^a so we can cancel it

$$\lambda^a \partial_c \mu_a = \lambda^a \mu_{a;c} + \mu_b \Gamma^b{}_{ac}\lambda^a$$

$$\mu_{a;c} = \partial_c \mu_a - \Gamma^b{}_{ac}\mu_b$$

Higher order: $\tau_{ab} = \lambda_a \mu_b$

$$\tau_{ab;c} = (\lambda_a \mu_b)_{;c} = \lambda_{a;c}\mu_b + \lambda_a \mu_{b;c}$$

$$= (\partial_c \lambda_a - \Gamma^d_{ac}\lambda_d)\mu_b + \lambda_a(\partial_c \mu_b - \Gamma^d_{bc}\mu_d)$$
$$= \partial_c(\lambda_a \mu_b) - \Gamma^d_{ac}\lambda_d \mu_b - \lambda_a \Gamma^d_{bc}\mu_d$$
$$= \partial_c \tau_{ab} - \Gamma^d_{ac}\tau_{db} - \Gamma^d_{bc}\tau_{ad}$$

This is exactly the same as the expression we got from taking the derivative of the metric tensor so $g_{ab;c} = 0$

Hence $\lambda_{a;c} = (g_{ab}\lambda^b)_{;c} = g_{ab;c}\lambda^b + g_{ab}\lambda^b{}_{;c} = g_{ab}\lambda^b{}_{;c}$ so raising and lowering indices can be done either before or after taking a covariant derivative.

4 Tensor and Physical Curvature

So we've seen that gravity=curvature. So it would be nice to have a way to quantify curvature. There is one (but it isn't nice!).

4.1 Riemann Curvature Tensor

Key thing about curvature is that neighbouring geodesics get further apart (or closer together) at a rate depending on the local curvature. Flat space, no forces, geodesics are straight lines. The distance between two geodesics then increases/decreases linearly as a function of path length along the geodesic. So the second derivative of the distance between them is zero. For curved space then the geodesics don't separate linearly - the second derivative will not be zero, and this gives us a way to quantify the curvature of spacetime.

Take two geodesics γ and $\tilde{\gamma}$ with coodinates $x^a(u)$ and $\tilde{x}^a(u)$. The distance between them is $\zeta^a(u) = \tilde{x}^a(u) - x^a(u) =$. These are geodesics so

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

$$\frac{d^2\tilde{x}^a}{du^2} + \tilde{\Gamma}^a_{bc}\frac{d\tilde{x}^b}{du}\frac{d\tilde{x}^c}{du} = 0$$

but

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \frac{\partial\Gamma^a_{bc}}{\partial x^d}(\tilde{x}^d - x^d) = \Gamma^a_{bc} + \partial_d\Gamma^a_{bc}\zeta^d$$

while subtracting the geodesics gives

$$\frac{d^2\tilde{x}^a}{du^2} + \tilde{\Gamma}^a_{bc}\frac{d\tilde{x}^b}{du}\frac{d\tilde{x}^c}{du} - \frac{d^2x^a}{du^2} - \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

$$\frac{d^2(x^a+\zeta^a)}{du^2} + (\Gamma^a_{bc} + \partial_d\Gamma^a_{bc}\zeta^d)\frac{d(x^b+\zeta^b)}{du}\frac{d(x^c+\zeta^c)}{du} - \frac{d^2x^a}{du^2} - \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

Let dot denote derivative with respect to path length, so this becomes

$$\ddot{x}^a + \ddot{\zeta}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a_{bc}\dot{x}^c\dot{\zeta}^b + \Gamma^a_{bc}\dot{\zeta}^b\dot{\zeta}^c +$$

$$\partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{\zeta}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^c\dot{\zeta}^b + \partial_d\Gamma^a_{bc}\zeta^d\dot{\zeta}^b\dot{\zeta}^c - \ddot{x}^a - \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$$

keep only first order terms in ζ and $\dot{\zeta}$ so ignore $\zeta^a\zeta^b$, $\zeta^a\dot{\zeta}^b$ and $\dot{\zeta}^a\dot{\zeta}^b$ - this is equivalent to saying the geodesics are originally close together. NB we keep the second derivative because if we had $y = x^2$ then $\dot{y} = 2x$ and $\ddot{y} = 2$. so if x is small then x and $\dot{x}$ are small but there is nothing you can do about 2!! so we keep second derivatives.

$$\ddot{\zeta}^a + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

We have this term $\ddot{\zeta}^a$. We are looking for a tensor equation, and we'd get something with this term that transformed as a tensor if we had $D^2\zeta^a/du^2$.

$$\frac{D^2\zeta^a}{du^2} = \frac{D}{du}(\dot{\zeta}^a + \Gamma^a_{bc}\zeta^b\dot{x}^c) = \frac{D}{du}\dot{\zeta}^a + \frac{D}{du}(\Gamma^a_{bc}\zeta^b\dot{x}^c)$$

$$= \ddot{\zeta}^a + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \frac{d}{du}(\Gamma^a_{bc}\zeta^b\dot{x}^c) + \Gamma^a_{ef}(\Gamma^e_{bc}\zeta^b\dot{x}^c)\dot{x}^f$$

$$= \ddot{\zeta}^a + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \frac{d\Gamma^a_{bc}}{du}\zeta^b\dot{x}^c + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \Gamma^a_{bc}\zeta^b\ddot{x}^c + \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f$$

We can substitute for $\ddot{x}$ from the geodesic equation along γ, and take the christoffel symbol derivative with respect to coordinates rather than the affine parameter

$$= \ddot{\zeta}^a + 2\Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^b\dot{x}^c\dot{x}^d - \Gamma^a_{bc}\zeta^b\Gamma^c_{ef}\dot{x}^e\dot{x}^f + \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f$$

so we then have an expression for $\ddot{\zeta}$ as

$$\ddot{\zeta}^a = \frac{D^2\zeta^a}{du^2} - 2\Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c - \partial_d\Gamma^a_{bc}\zeta^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\zeta^b\Gamma^c_{ef}\dot{x}^e\dot{x}^f - \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f$$

Substitute for $\ddot{\zeta}^a$ into our original equation for the separation of two geodesics

$$\ddot{\zeta}^a + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

$$\frac{D^2\zeta^a}{du^2} - 2\Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c - \partial_d\Gamma^a_{bc}\zeta^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\zeta^b\Gamma^c_{ef}\dot{x}^e\dot{x}^f - \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c$$
$$+\Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

$$\frac{D^2\zeta^a}{du^2} - \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c - \partial_d\Gamma^a_{bc}\zeta^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\zeta^b\Gamma^c_{ef}\dot{x}^e\dot{x}^f - \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

But the Christoffel symbols are symmetric so $\Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c = \Gamma^a_{cb}\dot{\zeta}^c\dot{x}^b = \Gamma^a_{bc}\dot{\zeta}^c\dot{x}^b$

$$\frac{D^2\zeta^a}{du^2} - \partial_d\Gamma^a_{bc}\zeta^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\zeta^b\Gamma^c_{ef}\dot{x}^e\dot{x}^f - \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

swap indices around on the last term $\partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = \partial_b\Gamma^a_{dc}\zeta^b\dot{x}^d\dot{x}^c$ so

$$\frac{D^2\zeta^a}{du^2} + (\Gamma^a_{be}\Gamma^e_{cd} - \partial_d\Gamma^a_{bc} - \Gamma^a_{ed}\Gamma^e_{bc} + \partial_b\Gamma^a_{dc})\zeta^b\dot{x}^c\dot{x}^d = 0$$

So here we have something which should equal zero if the space is flat, and not zero if the space is curved. This can all be stuck together into the Riemann curvature tensor

$$\frac{D^2\zeta^a}{du^2} + R^a_{cbd}\zeta^b\dot{x}^c\dot{x}^d = 0$$

where

$$R^a_{cbd} = (\Gamma^a_{be}\Gamma^e_{cd} - \partial_d\Gamma^a_{bc} - \Gamma^a_{ed}\Gamma^e_{bc} + \partial_b\Gamma^a_{dc})$$

Geodesics in flat space maintain their separation. Those in curved space do not.