1 Introduction to general relativity

1.1 Admin

Recommended reading is Foster & Nightingale 'A Short Course in General Relativity', Springer. I also use Schutz 'A first course in General Relativity', CUP, but I also put all my lecture notes on the web. For general reading (just for fun) try the Uncle Albert series by Russell Stannard, and Brian Greene 'The Elegant Universe'. These give you some nice ways to think about GR.

1.2 Special Relativity

Einsteins special relativity was a towering intellectual achievement, disposing of the 'intutively obvious' ideas of space and time as separate entities. Time does not flow evenly, the same for everyone, and spatial distances are not identical for all observers. The absolute, invariant thing is not space or time but the spacetime interval

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = g_{\mu\nu} dx^\mu dx^\nu$$

1.3 Acceleration

Special relativity is an amazing intellectual achievement. But there is a pretty severe limitation in this - special relativity only describes behaviour in INERTIAL FRAMES. So how do we treat acceleration, and describe behaviour in accelerated frames ?

Lets think about some accelerated frames. Circular motion is a nice way to get an accelerated frame since the speed stays constant. So think of a spinning roundabout. One person crawls around ON THE GROUND with a ruler, and lays it end to end to measure the circumference. Another person crawls underneath it to measure the diameter. The ratio circumference/diameter is then simply pi.

But what does someone ON the roundabout measure ? If the ruler is small enough then the ruler lies along the direction of motion, so we see its length is shortened. so the person on the roundabout needs more rulers to get around the circumference than the person who crawled around the outside circumference who was at aproximately rest with respect to us. (yes, there are some sublties, but see the notes at the back of The Elegant Universe). But along the radius the ruler is perpendicular to the direction of motion so there is no length contraction so the diameter comes out to be the same as before. So the ratio circumference/diameter is BIGGER than π. There is NO WAY to do this in flat space! If we go onto the surface of a sphere (the surface falls away from us in both directions - positively curved space) then the ratio circumference/diameter is SMALLER than pi (think of the limit of a circle of radius equal to that of the sphere - the circumference is $2\pi r$ but the diameter is πr so the ratio is 2!). If we go to the inside of a sphere we are still in positively curved space - it curves towards us in two directions. But we CAN do it in negatively curved space - like a saddle shape where it curves away from us in one direction and towards us the other.

So, if we are to handle acclerating frames we MUST use curved space. And since we know from special relativity that time is also a dimension we have to handle curved spacetime. Nasty nasty nasty. So we have to do some fairly advanced maths in order to figure out path lengths in curved space - its called differential geometry and its a bit grim.

1.4 Acceleration=gravity: Principle of equivalence

So do we REALLY want to handle accelerated frames - can't we just stick to inertial frames ? Well, no, because we want to understand gravity. And Newtons gravity is in big trouble as it has a subtle but direct conflict with SR. Newtons theory of gravity is amazingly accurate in calculating the effects of gravity in the world around us, but completely silent on what causes gravity. In Newtons formulae, gravity is solely determined my the mass and separation of objects. Changing the mass or separation then, according to Newton, INSTANTLY changes the gravity. But special relativity says NOTHING can travel faster than light. Einstens 'happiest thought' was the realisation that there was no difference between gravity and an accelerating frame - there is no difference between an accelerated frame without a gravitational field and a stationary frame within a gravitational field. Or, say the same thing a different way around, there is no difference between a free fall frame in a gravitational field and an inertial frame.

This is the equivalence principle. *In a freely falling (nonrotating) frame then the laws of physics are the laws of special relativity*

So there is a deep connection between gravity and accelerated motion.

This explains why the inertial and gravitational masses are the same, which was a big deep problem. Inertial mass, m_i, is the one which gives the constant of proportionality between force and acceleration - Newtons famous $F = m_i a$. Gravitational mass, m_g, determines how the object is affected by the force of gravity $F = GMm_g/r^2 = m_g g$. So when an object falls under a gravitational field we have $a = m_g/m_i g$. The only way for all objects to fall the same way in a gravitational field (feather and hammer) is if $m_g/m_i =$ constant. If acceleration and gravity are two different things then this is deeply strange. Compare it with electrostatics - a test particle of charge q and inertial mass m_i in a field E experiences a force qE and acceleration $m_i a$. So $a = q/m_i E$ and different particles have difference values of q/m_i. But they don't have different values of m_g/m_i! But if gravity is the same as being accelerated then its not at all strange. But acceleration REQUIRED us to think about curved spacetime, so we need to do differential geometry in order to understand gravity!!!

1.5 Curved space

Think of ants with ink on their feet, walking normally i.e. taking steps of equal length with their left and right feet. On a flat piece of their inky feet leave a path which is a straight line. On a curved surface, still taking steps of equal length, they trace out a curved path. e.g. two ants setting off from different points along the equator of a sphere, heading due south. Their paths get closer and closer together AS IF there were an attractive force between them. But there is not, its just that they are travelling normally (steps of equal length with left and right feet) over a surface that is curved. Now put the ants on the surface of an apple. they still move 'normally' but now their paths show a variety of curves depending on the curvature of the surface of the apple - if the ants walk close to the stalk where the surface is highly curved then their path curves a lot.

These 'walking normally' paths in physical language are constant velocity frames - i.e. inertial frames. But in geometric language they are geodesic paths - shortest distance between two points.
1) locally geodesics (shortest distance between two points) appear straight
2) over more extended regions of spacetime then geodesics originally receeding from each other begin to approach at a rate governed by the curvature of spacetime, and this effect of geometry on matter is what we mean today

by the old word gravitation
3) matter in turn warps geometry

Space tells matter how to move, matter tells space how to curve

1.6 The way ahead

1) figure out how mass(energy) curves spacetime(!!!)
2) find the geodesics of our curved spacetime - ie the inertial frames. we know how to do physics here (Special relativity)
3) write the physical laws in such a way that we can easily transform the equations to any arbitrary frame (not just inertial ones). In these frames the paths will CURVE, AS IF THERE WAS SOME ADDITIONAL FORCE ACTING. But its because we have transformed to a 'bad' coordinate system, where the curvature of spacetime is apparent. This is where tensors come in.

1.7 Gravity and light: The Pounds-Rebka-Snider experiment

This was a Gedanken (thought) experiment first done by Einstein. In the old way to think of gravity, then only particles with mass are acted on by gravity, so photons, being massless, are not. Which leads to some horrors like infinite energy!

Think of a particle dropped from a tower of height h. Its energy when it reaches the ground is $m_0c^2 + m_0gh$. Convert it into a photon then it has $h\nu_{bottom} = m_0c^2 + m_0gh$. If gravity doesn't affect light then it gets to the top with this energy so $h\nu_{top} = m_0c^2 + m_0gh$, to be converted back into a particle. But now the particle mass is $m = m_0 + m_0gh/c^2$. So we can drop it again..... Since we believe in conservation of energy then we'd better have gravity affecting light, giving a gravitational redshift so that $h\nu_{top} = m_0c^2$. Hence

$$\frac{h\nu_{top}}{h\nu_{bottom}} = \frac{m_0c^2}{m_0c^2 + m_0gh} = 1 - gh + ...$$

And in fact we can see this from the roundabout again - fast clocks run slow in SR so a clock in an accelerating frame run slow, ie gravitational redshift/time dilation

And we can tie in the equivalence principle as well. Suppose we had our energy to mass converter in a lab which was freely falling such that it arrived at the top of the tower from above just as the photon arrived at the top of the tower from below. The lab is then moving with velocity gh downwards, so the gravitationally redshifted photon is doppler blueshifted with respect to the lab. And the effects cancel (to first order as appropriate for these approximations) so *there is no redshift measured in the free fall frame*. So a free fall frame really is an inertial frame!

2 Mathematical Background

2.1 Introduction

So, we have to do curved spacetime. There are some very powerful mathematical techniques for handling arbitrary curved spaces. They are nasty, but we want to understand gravity, so we have to persevere. This gets us into differential geometry, of which we will do AS MUCH AS WE NEED (no more, no less) to do gravity. We want to be able to characterise the curvature of spacetime which is associated with the presence of mass-energy.

Another separate thing we need is to be able to write equations so that we can transform easily between frames (geodesic paths in curved spacetime are inertial frames. We can write down the equations of physics ie the equations of special relativity in these inertial frames, then transform coordinates out of these into an arbitrary frame and we automatically account for gravity!). Turns out we can do BOTH curvature and coordinate transformations of equations by using TENSORS.

2.2 Notation

First lets start with standard vectors $\vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in standard cartesian coordinates. This notation is incredibly clumsy. So we're going to use instead a much more powerful and very compact notation. Axes are labeled by indices so $\{x, y, z\} = \{x^1, x^2, x^3\}$ where the superscripts DO NOT MEAN 'raised to the power' but instead pick out the components along the x, y and z axes i.e. along the **i,j,k** directions which we are now going to call $\vec{e_1}, \vec{e_2}, \vec{e_3}$. If we need to raise to the power we will make it obvious e.g $(x^1)^3$. Then we can write this as

$$\vec{r} = x^1\vec{e_1} + x^2\vec{e_2} + x^3\vec{e_3}$$

It doesn't look much better yet. But we are going to get very sick of writing all the coordinates so we're going to use shorthand notation that $\{x^i\}$ means all of the $\{x^1, x^2, x^3\}$. We're also going to use the Einstein summation convection: whereever an expression contains one index as a superscript and the SAME index as a subscript then the summation is implied. So our vector

$$\vec{r} = x^1\vec{e_1} + x^2\vec{e_2} + x^3\vec{e_3} = x^i\vec{e_i} = x^j\vec{e_j}$$

It really doesn't matter what we call the index we sum over - if it occurs in the sum, we sum over it, whatever it is! But we'll have a convention that roman letters from the middle of the alphabet go from 1-3 (3D space, i,j...), while roman letters from the start of the alphabet (a,b,c...) mean $\{x^1, x^2..x^N\}$ - any N dimensional space and capital roman letters from the start of the alphabet (A,B....) mean 2D space. Greek letters are for spacetime, and go from 0-3, so $\{x^0, x^1, x^2, x^3\}$ where $x^0 = ct$, and $\{x^i\} = \{x, y, z\}$.

Once we have picked a given space(time) then we might do coordinate transformations within this. Up till now you've often seen this denoted as the primed coordinate frame, with the prime on the kernal letter, e.g. $x' = \gamma(x - vct/c)$ as the Lorentz transformation for the x coordinate I'm going to put a bar on the *index* to denote a transformation to a different frame (a prime can all to easily get mistaken for a 1, and I'm going to do it on the individual indices rather than on the kernal letter as we're going to generalise to things with more than 1 index...!!) so it looks like $x^{\overline{1}} = \gamma(x^1 - vx^0/c)$.

2.3 Generalised coordinates in N dimensions

Lets now think about more general spaces which have arbitrary curvature. Define a point P in some space, and another point a little further on called R. These points have NOTHING to do with the coordinate system. They exist irrespective of the labels we give them. Coordinates just make them easier to handle. So lets take two points, P and R, and look at the displacement vector which points from one event to another whose components are the coordinate differences. A key property of tensors is that their representations in different coordinate systems depend only on the relative orientations and scales on the coordinate axes at that point and NOT on the absolute values of coordinates. So thats why we used the differential position vector not the absolute position vector from the origin to a point. This VECTOR, connecting them, $\vec{r}$, will be the same in all coordinate systems, though its components will be different depending on our choice of coordinates.

$$\vec{r} = \Delta x^1 \vec{e_1} + \Delta x^2 \vec{e_2} + ... = \Delta x^a \vec{e_a}$$

is an arrow pointing from P to R, with components Δx^a along whatever basis vectors $\vec{e_a} = \partial \vec{r}/\partial x^a$ we are using (they don't have to be rectilinear or orthoganal).

Now transform to a different coordinate system - the vector is the SAME vector. We can write the new (primed) coordinates, as function of the old (unprimed) ones so $\Delta x^{\overline{a}} = \Delta x^{\overline{a}}(\Delta x^b)$. (our conventions mean that this expands to N equations):

$$\Delta x^{\overline{1}} = \Delta x^{\overline{1}}(\Delta x^1, \Delta x^2 .... \Delta x^N)$$
$$\Delta x^{\overline{2}} = \Delta x^{\overline{2}}(\Delta x^1, \Delta x^2 ... \Delta x^N)$$

etc... But we know how to do total differentials:

$$\Delta f = \frac{\partial f}{\partial x^1}\Delta x^1 + \frac{\partial f}{\partial x^2}\Delta x^2 + ... \frac{\partial f}{\partial x^N}\Delta x^N = \frac{\partial f}{\partial x^a}\Delta x^a$$

so we can write our coordinate transformations as (the N equations!)

$$\Delta x^{\overline{b}} = \frac{\partial x^{\overline{b}}}{\partial x^a}\Delta x^a$$

where there are N^2 separate $\partial x^{\overline{b}}/\partial x^a$. In the limit of infinitesimals this goes to

$$dx^{\overline{b}} = \frac{\partial x^{\overline{b}}}{\partial x^a} dx^a$$

2.4 Example from SR - The Lorentz transformation

And we KNOW these coordinate transformations - they are the Lorentz transformations for a standard boost along the x axis

$$x^{\overline{0}} = \gamma(x^0 - vx^1/c) \quad x^{\overline{1}} = \gamma(x^1 - vx^0/c) \quad x^{\overline{2}} = x^2 \quad x^{\overline{3}} = x^3$$

So now lets take differentials

$$\frac{\partial x^{\overline{0}}}{\partial x^0} = \gamma \quad \frac{\partial x^{\overline{0}}}{\partial x^1} = -\gamma v/c \quad \frac{\partial x^{\overline{0}}}{\partial x^2} = 0 \quad \frac{\partial x^{\overline{0}}}{\partial x^3} = 0$$

$$\frac{\partial x^{\overline{1}}}{\partial x^0} = -\gamma v/c \quad \frac{\partial x^{\overline{1}}}{\partial x^1} = \gamma \quad \frac{\partial x^{\overline{1}}}{\partial x^2} = 0 \quad \frac{\partial x^{\overline{1}}}{\partial x^3} = 0$$

$$\frac{\partial x^{\overline{2}}}{\partial x^0} = 0 \quad \frac{\partial x^{\overline{2}}}{\partial x^1} = 0 \quad \frac{\partial x^{\overline{2}}}{\partial x^2} = 1 \quad \frac{\partial x^{\overline{2}}}{\partial x^3} = 0$$

$$\frac{\partial x^{\overline{3}}}{\partial x^0} = 0 \quad \frac{\partial x^{\overline{3}}}{\partial x^1} = 0 \quad \frac{\partial x^{\overline{3}}}{\partial x^2} = 0 \quad \frac{\partial x^{\overline{3}}}{\partial x^3} = 1$$

so we can write this in our compact summation notation as

$$x^{\overline{\alpha}} = \frac{\partial x^{\overline{\alpha}}}{\partial x^\beta} x^\beta = \Lambda^{\overline{\alpha}}_\beta x^\beta$$

A superscript index on the denominator counts the same in the summation convention as a subscript on the numerator.

2.5 Contravariant tensors of 1st order (4 vectors)

Entities which transform like the coordinate differences are called contravariant tensors of first order, or (1,0) tensors. They are defined by their transformation properties. If $\vec{A} = A^a \vec{e_a}$ has components which transform as

$$A^{\overline{b}} = \frac{\partial x^{\overline{b}}}{\partial x^a} A^a$$

then it is a contravariant tensor

2.6 More examples from SR

In spacetime then the 4-vectors you have met already are contravarient tensors e.g. the 4-velocity, $U^\alpha = m dx^\alpha / d\tau$ must transform like the coordinate differences (dx^α is a coordinate difference, $d\tau$ has nothing to do with coordinates - its an invarient).

But to see this explicitally when we look in another (primed) coordinate frame and we get

$$U^{\overline{\alpha}} = \frac{dx^{\overline{\alpha}}}{d\tau} = \frac{\partial x^{\overline{\alpha}}}{\partial x^\beta} \frac{dx^\beta}{d\tau} = \frac{\partial x^{\overline{\alpha}}}{\partial x^\beta} U^\beta$$

Any tensor equation (an equation involving these tensors and their derivatives with respect to an invarient) can be made to look the same irrespective of the coordinate system. e.g. the 4-force $f^\alpha = m_0 dU^\alpha/d\tau$ - its the SR equivalent of Newtons $\vec{F} = m\vec{a}$. Now go to a different frame and if this is a 1st order covarient tensor then

$$f^{\overline{\beta}} = \frac{\partial x^{\overline{\beta}}}{\partial x^\alpha} f^\alpha$$

but the other side of the equation is now

$$m_0 \frac{dU^{\overline{\beta}}}{d\tau} = \frac{d}{d\tau} \frac{\partial x^{\overline{\beta}}}{\partial x^\alpha} U^\alpha$$

Since the partial derivatives depend on coordindates, not (invarient) proper time then differentiation gives

$$m_0 \frac{\partial x^{\overline{\beta}}}{\partial x^\alpha} \frac{dU^\alpha}{d\tau} = \frac{\partial x^{\overline{\beta}}}{\partial x^\alpha} f^\alpha$$

So the equation looks the same in any frame.

2.7 What about the basis vectors?

Back to the coordinate differences

$$\vec{r} = \Delta x^a \vec{e_a} = \Delta x^{\overline{b}} \vec{e_{\overline{b}}}$$

Our new basis vectors are again the tangents to the coordinate curves so these are

$$e_{\overline{b}} = \frac{\partial \vec{r}}{\partial x^{\overline{b}}} = \frac{\partial x^a}{\partial x^{\overline{b}}} \vec{e_a}$$

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This is a bit different. Our basis vectors don't transform like the coordinate differences i.e. like contravarient tensors which have $A^{\overline{b}} = \partial x^{\overline{b}}/\partial x^a A^a$ -

they go the other way (derivative of the old with respect to the new, whereas the components of the vectors - contravarient tensors of first order - transformed as the derivatives of the new with respect to the old). So this looks more like a INVERSE transformation if we were talking about coordinates. Lets look at this in some more detail by looking at the inverse transformations (from primed to unprimed frame). They are done in much the same way as the transformations we wrote before from the unprimed to primed frames.

$$\Delta x^a = \Delta x^a(\Delta x^{\overline{b}})$$
$$\Delta x^a = \frac{\partial x^a}{\partial x^{\overline{b}}} \Delta x^{\overline{b}}$$

or in the infinitesimal limit

$$dx^a = \frac{\partial x^a}{\partial x^{\overline{b}}} dx^{\overline{b}}$$

and yes, we can see that the change in basis vectors from the unprimed to primed frame goes like the coordinate differences from the primed to unprimed frame! So they don't transform like the coordinate differences, they transfrom like the INVERSE of the coordinate differences.

Lets see this from another viewpoint. We can write any contravarient 1st order tensor $\vec{A}$ as

$$\vec{A} = A^a \vec{e_a} = A^{\overline{b}} \vec{e_{\overline{b}}}$$

But we know how contravarient tensor components transform - its $A^{\overline{b}} = \partial x^{\overline{b}} / \partial x^a A^a$. But we can just rename indices and equally well write this as $A^a = \partial x^a / \partial x^{\overline{c}} A^{\overline{c}}$ so

$$A^{\overline{c}} \frac{\partial x^a}{\partial x^{\overline{c}}} \vec{e_a} = A^{\overline{b}} \vec{e_{\overline{b}}}$$

change dummy indices to whatever you like

$$A^{\overline{b}} \frac{\partial x^a}{\partial x^{\overline{b}}} \vec{e_a} = A^{\overline{b}} \vec{e_{\overline{b}}}$$

$$A^{\overline{b}} \left(\frac{\partial x^a}{\partial x^{\overline{b}}} \vec{e_a} - \vec{e_{\overline{b}}} \right) = 0$$

$$\vec{e_{\overline{b}}} = \frac{\partial x^a}{\partial x^{\overline{b}}} \vec{e_a}$$

and we get the same answer (phew!). So this is the way the basis vectors themselves change - its is NOT a component transformation. It gives the new *basis vectors* (not components) $\vec{e_{\overline{b}}}$ in terms of the partial differentials of the old coordinates with respect to the new, whereas the contravariant tensors we looked at before transformed as the partial differentials of the new coordinates wrt to the old.

2.8 Covariant tensors of first order

Lets go back into 3D and look at how the components of a scalar field $\phi(x^i)$ transform. Scalar fields are just numbers at a given point. whatever coordinates you give the point makes no change in the number. But its gradient will change depending on coordinate system (because we take gradients as change in number over some coordiante distance). Its gradient is simply $\nabla\phi(x^i)$ which has components

$$\frac{\partial\phi}{\partial x^i}$$

so go to another frame with primed coordindates and look at $\nabla\phi(x^{\overline{j}})$ which has components

$$\frac{\partial\phi}{\partial x^{\overline{j}}} = \frac{\partial\phi}{\partial x^i}\frac{\partial x^i}{\partial x^{\overline{j}}}$$

so the gradient is NOT a contravariant tensor. So there are physical quantities we will need to describe which are NOT contravarient. Gradients have components which transform the other way to the way a contravariant tensor transforms. Goes as old wrt new not new wrt old i.e. like the normal basis vectors. We call things that transform like this COVARIANT vectors (also called (0,1) tensors or covariant tensors of the first kind/order, or one forms) and we denote these by a lower index on the components (like the normal basis vectors are denoted by a lower index), and transformation laws for the components (now in generalised N dimensions) are:

$$A_{\overline{b}} = \frac{\partial x^a}{\partial x^{\overline{b}}} A_a$$

2.9 Metric tensor

Lets go back to our generalised differential position vector again

$$\vec{\Delta x} = \Delta x^a \vec{e_a}$$

This is telling us the distance between two points. Suppose in fact we're looking at a curve in this N dimensional space, where position along the curve can be parameterized as some parameter t

$$\vec{r} = x^a(t)\vec{e_a}$$

$$\vec{\Delta r} = \frac{\partial \vec{r}}{\partial x^b}\frac{dx^b}{dt}\Delta t \vec{e_b}$$

$$\frac{\vec{dr}}{dt} = \frac{dx^b}{dt}\vec{e_b}$$

So the length of the curve is

$$L = \int_{t1}^{t2} |\vec{dr}/dt.\vec{dr}/dt|^{1/2} dt$$

or in infinitesimals

$$ds^2 = \vec{dr}.\vec{dr}$$

$$= dx^a \vec{e_a}.dx^b \vec{e_b}$$

$$= \vec{e_a}.\vec{e_b} dx^a dx^b$$

$$= g_{ab} dx^a dx^b$$

where $g_{ab} = \vec{e_a}.\vec{e_b}$ is called the metric. Its transformation under coordinate change can be seen as we derived the basis vector transformations

$$\vec{e_{\overline{a}}}.\vec{e_{\overline{b}}} = \frac{\partial x^c}{\partial x^{\overline{a}}}\vec{e_c}.\frac{\partial x^d}{\partial x^{\overline{b}}}\vec{e_d}$$

$$= \frac{\partial x^c}{\partial x^{\overline{a}}}\frac{\partial x^d}{\partial x^{\overline{b}}}\vec{e_c}.\vec{e_d}$$

$$= \Lambda^c_{\overline{a}}\Lambda^d_{\overline{b}} g_{cd}$$

So the components transform like the basis vectors twice - called covariant tensor of second order - this is the METRIC tensor and its wildly important in everything that we are going to do next, because it includes what happens to distances (spacetime intervals!!) if the space is curved.

From its definition in terms of the dot product of basis vectors then we can see that $g_{ab} = g_{ba}$, ie that it is symmetric.

The metric tensor is basically a rule for taking two vectors and getting a single number out.

2.10 Example metric for the surface of a sphere

I've said that the machinary we've developed is very general - we can use it on spaces of arbitrary curvature as well as flat Euclidean spaces. So lets see it at work on a CURVED surface - the surface of a sphere. Here we are embedding a 2D curved surface into a 3D Euclidean geometry. So we have $\vec{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=x^i\vec{e_i}$. Now transform coordinates to the surface of a sphere of radius a where a point can be parameterised in terms of θ and ϕ on the surface where $x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$. so now we can write $\vec{r} = x^{\overline{A}}e_{\overline{A}}$ where I'm using the convention that capital roman letters mean a 2D space. So lets write the coordinate trasnformation as primed coordinates to make it all look like the stuff we've been dealing with so

$$x^1 = a\sin x^{\overline{1}}\cos x^{\overline{2}} \quad x^2 = a\sin x^{\overline{1}}\sin x^{\overline{2}} \quad x^3 = a\cos x^{\overline{1}}$$

the new basis vectors (the ones along the surface of the sphere) are

$$e_{\overline{A}} = \frac{\partial\vec{r}}{\partial x^{\overline{A}}} = \frac{\partial x^i}{\partial x^{\overline{A}}}\vec{e_i}$$

so for the $x^{\overline{1}}$ i.e. θ coordinate we have

$$e_{\overline{1}} = \frac{\partial x^1}{\partial x^{\overline{1}}}\vec{e_1} + \frac{\partial x^2}{\partial x^{\overline{1}}}\vec{e_2} + \frac{\partial x^3}{\partial x^{\overline{1}}}\vec{e_3} = a\cos x^{\overline{1}}\cos x^{\overline{2}}\vec{e_1} + a\cos x^{\overline{1}}\sin x^{\overline{2}}\vec{e_2} - a\sin x^{\overline{1}}\vec{e_3}$$

or more simply $\vec{e_\theta} = a\cos\theta\cos\phi\mathbf{i}+a\cos\theta\sin\phi\mathbf{j}-a\sin\theta\mathbf{k}$. Similarly

$$e_{\overline{2}} = \frac{\partial x^1}{\partial x^{\overline{2}}}\vec{e_1} + \frac{\partial x^2}{\partial x^{\overline{2}}}\vec{e_2} + \frac{\partial x^3}{\partial x^{\overline{2}}}\vec{e_3} = -a\sin x^{\overline{1}}\sin x^{\overline{2}}\vec{e_1} + a\sin x^{\overline{1}}\cos x^{\overline{2}}\vec{e_2}$$

or more simply $\vec{e_\phi} = -a\sin\theta\sin\phi\mathbf{i}+a\sin\theta\cos\phi\mathbf{j}$.

The metric tensor components are the dot products of the basis vectors so

$$g_{\overline{11}} = \vec{e_{\overline{1}}}.\vec{e_{\overline{1}}} = a^2\cos^2 x^{\overline{1}}\cos^2 x^{\overline{2}} + a^2\cos^2 x^{\overline{1}}\sin^2 x^{\overline{2}} + a^2\sin^2 x^{\overline{1}} = a^2$$

$$g_{\overline{12}} = \vec{e_{\overline{1}}}.\vec{e_{\overline{2}}} = -a^2\cos x^{\overline{1}}\cos x^{\overline{2}}\sin x^{\overline{1}}\sin x^{\overline{2}} + a^2\cos x^{\overline{1}}\sin x^{\overline{2}}\sin x^{\overline{1}}\cos x^{\overline{2}} = 0$$

$$g_{\overline{22}} = \vec{e_{\overline{2}}}.\vec{e_{\overline{2}}} = a^2\sin^2 x^{\overline{1}}\sin^2 x^{\overline{2}} + a^2\sin^2 x^{\overline{1}}\cos^2 x^{\overline{2}} = a^2\sin^2 x^{\overline{1}}$$

So the line element showing us how distance relates to changing coordinates on the surface of a sphere is the familiar one

$$ds^2 = g_{\overline{AB}}dx^{\overline{A}}dx^{\overline{B}} = a^2(dx^{\overline{1}})^2 + a^2\sin^2 x^{\overline{2}}(dx^{\overline{2}})^2 = a^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

Well, its familar if you can remember anything about spherical polar coordinates!

Here we have used our knowledge of the curvature of space to define the metric. Alternatively we can of course go the other way and use the metric to DEFINE how distance between points relates to coordinates. We've already seen one metric tensor which embodies flat spacetime, that of special relativity where $g_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$. Its is NOT positive definite. $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. In mathematical terms, a manifold is some N dimensional space which can be characterised by a metric tensor (ie coordinates can be related to distance and its differentiable). Reimanian manifolds have positive definite distance, pseudo-Reimanian do not. So the familar spacetime can be terms 'a 4D pseudo-Reimanian manifold'. I'll just stick to spacetime.

2.11 Basis vectors for covariant components

We got the basis vectors for the contravariant components by getting the vectors tangent to the coordinate curves. We've been working with these basis vectors which define a tangent space T_P at P. A contravariant vector $\vec{\lambda} = \lambda^a\vec{e_a}$ is an arrow emmanating from P - its not in the manifold like the curve is in the manifold, but is attached to it like the tangent space is attached to a surface.

But we could instead have defined a set of basis vectors by the vectors which were normal to the coordinate surfaces rather than tangent to the coordinate curves. We get a coordinate surface by holding that coordinate constant. and then we want to get the normal to this surface. We get normals by taking the GRADIENT. We can regard each coordinate as a scalar field Then the vector perpendicular to this scalar field is $\vec{e^a} = \nabla x^a$. And if we do a transformation of coordinates $x^{\overline{b}} = x^{\overline{b}}(x^a)$ then these basis components transform as

$$\vec{e^{\overline{b}}} = \nabla x^{\overline{b}} = \frac{\partial x^{\overline{b}}}{\partial x^a} \nabla x^a = \frac{\partial x^{\overline{b}}}{\partial x^a} \vec{e^a}$$

i.e. these transform in the opposite way to our tangent basis vectors.

This gives us another vector space of the manifold - call it T_P^* the dual or cotangent space of the manifold at P. The basis vectors themselves look identical if we have an orthonormal set of coordinates (they don't have to be reclilinear, just be 90 degrees where they meet). BUT THEY ARE NOT IDENTICAL IF THE COORDINATES ARE NOT ORTHOGANAL.

2.12 Covariant and contravariant: more on the metric

But if we have another set of basis vectors then we can write any arbitrary vector either on the old basis in the tangent space OR the new basis in the cotangent space i.e. $\vec{\lambda} = \lambda^a \vec{e_a} = \lambda_b \vec{e^b}$. If the basis vectors are the same i.e. we had orthonormal bases then the contravariant and covariant components are IDENTICAL. But of course in general they are not.

For the pure mathematically inclined, setting the contravariant and covariant forms of writing this as equal is a bit of a fudge as $\lambda^a \vec{e_a}$ and $\lambda_b \vec{e^b}$ are actually are in two different vector spaces T_P and $T*_P$, respectively. But we'll do a bit more on this later

Take another vector $\vec{\mu} = \mu^a \vec{e_a} = \mu_a \vec{e^a}$. The dot product of these is

$$\vec{\lambda}.\vec{\mu} = \lambda^a \vec{e_a}.\mu^b \vec{e_b} = \lambda^a \mu^b \vec{e_a}.\vec{e_b} = \lambda^a \mu^b g_{ab}$$

But we could have done this with the covariant components alone

$$\vec{\lambda}.\vec{\mu} = \lambda_a \vec{e^a}.\mu_b \vec{e^b} = \lambda_a \mu_b \vec{e^a}.\vec{e^b} = \lambda_a \mu_b g^{ab}$$

where we have defined $g^{ab} = \vec{e^a}.\vec{e^b}$. Again this is symmetric by definition of the dot product so $g^{ab} = g^{ba}$. But we could also have done this with the mixed components

$$\vec{\lambda}.\vec{\mu} = \lambda^a \vec{e_a}.\mu_b \vec{e^b} = \lambda^a \mu_b \vec{e_a}.\vec{e^b}$$

What is this dot product ? lets do it explicitally in 3D

$$\vec{e_i}.\vec{e^j} = \frac{\partial \vec{r}}{\partial x^i}.\nabla x^j$$

where $\vec{r} = x^i \vec{e_i}$. Then we have

$$e_i.e^j = (\frac{\partial x^1}{\partial x^i} i + \frac{\partial x^2}{\partial x^i} j + \frac{\partial x^3}{\partial x^i} k).(\frac{\partial x^j}{\partial x^1} i + \frac{\partial x^j}{\partial x^2} j + \frac{\partial x^j}{\partial x^3} k)$$

$$= \frac{\partial x^1}{\partial x^i}\frac{\partial x^j}{\partial x^1} + \frac{\partial x^2}{\partial x^i}\frac{\partial x^j}{\partial x^2} + \frac{\partial x^3}{\partial x^i}\frac{\partial x^j}{\partial x^3}$$

$$= \frac{\partial x^j}{\partial x^i} = \delta_i^j$$

where the kronekar delta function is $\delta_i^j = 1$ for $i = j$, 0 otherwise. NB because of the summation convention then $\delta_i^i = \delta_1^1 + \delta_2^2 + \delta_3^3 = 3$ So now we have

$$\lambda^a \vec{e_a}.\mu_b \vec{e^b} = \lambda^a \mu_b \vec{e_a}.\vec{e^b} = \lambda^a \mu_b \delta_a^b = \lambda^a \mu_a$$

By symmetry of the dot product and metric then $\lambda^a \mu_a = \lambda_a \mu^a$. So collecting all these together we have

$$\lambda^a \mu_a = \lambda_a \mu^a = g_{ab} \lambda^a \mu^b = g^{ab} \lambda_a \mu_b$$

So this gives us an easy way of swapping between contravariant and covariant components - we use the METRIC $g_{ab}\lambda^a \mu^b = \lambda_a \mu^a$ so

$$\lambda_a = g_{ab} \lambda^b$$

and as $g^{ab}\lambda_a \mu_b = \lambda^b \mu_b$ so

$$\lambda^a = g^{ab} \lambda_b$$

Tensors derived from other tensors by raising or lowering the indices via the metric are called **associated tensors**.

We also get another useful metric relation in that $\lambda_a = g_{ab}\lambda^b = g_{ab} g^{bc} \lambda_c$ so

$$g_{ab} g^{bc} = \delta_c^a$$

For the pure mathematically inclined, heres how its done. A contravariant vector $\vec{\lambda} = \lambda^a \vec{e_a}$ in T_p is associated with a covariant vector $\tilde{\lambda} = \lambda_a \vec{e^a}$ in $T{*}_P$

via the metric (or dot or inner product) such that for all vectors $\vec{\mu}$ in T_P then $\tilde{\lambda}(\vec{\mu})$ is a real number $= \mu^a \lambda_a$. i.e. it define covariant vectors as linear functions which map covariant vectors (one–forms) into real numbers via the metric. Schutz takes this approach (eventualy), as do Misner, Thorne and Wheeler. Foster and Nightingale take the approach I've used here.

2.13 Summary: Tensor definitions and algebra

Suppose that at a point P on a manifold M there are N^{r+s} quantities $\tau^{a_1...a_r}_{b_1...b_s}$ which under a change of coordinates transform as

$$\tau^{\overline{a_1}...\overline{a_r}}_{\overline{b_1}...\overline{b_s}} = \frac{\partial x^{\overline{a_1}}}{\partial x^{c_1}} ... \frac{\partial x^{\overline{a_r}}}{\partial x^{c_r}} \frac{\partial x^{d_1}}{\partial x^{\overline{b_1}}} ... \frac{\partial x^{d_s}}{\partial x^{\overline{b_s}}} \tau^{c_1...c_r}_{d_1...d_s}$$

where the partial derivatives of the coordinates are also evaluated at P. Then the quantities $\tau^{a_1...a_r}_{b_1...b_s}$ are called components of as type (r,s) tensor at P. The sum (r+s) is called the rank or order of the tensor, (r,0) might be called a contravariant tensor of rank r, while a (0,s) might be called a covariant tensor of rank s. If both r and s are nonzero then its called a mixed tensor. scalars are type (0,0) tensors. The whole point is that tensor equations maintain physical meaning under coordinate transformation.

linear combination

$$T^c_d = aA^c_d + bB^c_d$$

where a and b are scalars and A^c_d, B^c_d are (1,1) mixed tensors. Then T^c_d is also a (1,1) mixed tensor.

$$T^{\overline{e}}_{\overline{f}} = aA^{\overline{e}}_{\overline{f}} + bB^{\overline{e}}_{\overline{f}} = a\frac{\partial x^{\overline{e}}}{\partial x^c}\frac{\partial x^d}{\partial x^{\overline{f}}}A^c_d + b\frac{\partial x^{\overline{e}}}{\partial x^c}\frac{\partial x^d}{\partial x^{\overline{f}}}B^c_d$$

$$= \frac{\partial x^{\overline{e}}}{\partial x^c}\frac{\partial x^d}{\partial x^{\overline{f}}}T^c_d$$

Direct (outer) Products We can form higher order tensors out of lower order ones by multiplying the components.

$$T^a_b = A^a B_b$$

Again, its easy to show that this transforms as required.

$$T^{\overline{a}}_{\overline{b}} = A^{\overline{a}} B_{\overline{b}} = \frac{\partial x^{\overline{a}}}{\partial x^c} A^c \frac{\partial x^d}{\partial x^{\overline{b}}} B_d = \frac{\partial x^{\overline{a}}}{\partial x^c} \frac{\partial x^d}{\partial x^{\overline{b}}} T^c_d$$

General Inner product (contraction)

Set an upper and lower index as equal i.e. summing it, gives a new tensor of rank 2 less than the original one. Generally we keep the same kernal letter e.g. if we have a tensor $T^{a\ cd}_{\ b}$ then $T^{ac} = T^{a\ cb}_{\ b}$. Though obviously this is not unique as we could have also had the DIFFERENT contractions $T^{ad} = T^{a\ bd}_{\ b}$ or $T^{cd} = T^{a\ cd}_{\ a}$

Raising and lowering indexes

This is simply a direct product of a tensor with the metric, followed by a contraction. We can use it to swap between covariant and contravariant components AND basis vectors so $A_a = g_{ab}A^b$, $\vec{e^a} = g^{ab}\vec{e_b}$ where $g^{ab}g_{bc} = \delta^a_c$

Inner Product of 2 vectors (dot or scalar product)

$$\vec{A}.\vec{B} = g_{ab}A^aB^b = A^aB_a = A_bB^b = A^1B_1 + A^2B_2 + ...A^NB_N$$

Because its a scalar it must be invarient. We can see this explicitally on a change of coordinates

$$A^{\overline{a}}B_{\overline{a}} = \frac{\partial x\overline{a}}{\partial x^c} A^c \frac{\partial x^d}{\partial x\overline{a}} B_d$$
$$= \delta^d_c A^c B_d = A^c B_c$$

An obvious and VERY important invarient scalar product is $ds^2 = c^2 d\tau^2 = dx^\alpha dx_\alpha = g_{\alpha\beta} dx^\alpha dx^\beta$

ANY EQUATION WILL BE INVARIANT UNDER GENERAL COORDINATE TRANSFORMATIONS IF IT STATES THE EQUALITY OF TWO TENSORS WITH THE SAME UPPER AND LOWER INDICES. $A^a_{bc} = B^a_{bc}$ can be written in another coordinate frame as $A^{\overline{a}}_{\overline{bc}} = B^{\overline{a}}_{\overline{bc}}$ since both sides transform between coordinate frames in the same way

3 Curved Space

We're going to assume that we'll always be dealing with a space which is continuous and differentiable - a differentiable manifold. We're also going to assume that it has a metric. If the metric is positive definite (inner products +ve) then this is called a Riemannian manifold. Our metric won't be - so called pseudo–Riemannian. We've seen that the metric can carry (unimportant) information about coordinates - if space is flat then there is always some coordinate choice which gets it back into the unit matrix, and that the metric can also carry important non–transformable away information about the space curvature e.g. the metric of SR spacetime and the metric's we'll be using to describe the curved spacetime of gravity.

So the whole idea is that there is no 'force-at-a-distance' gravity. Paths curve only because they are following locally curved spacetime (which is curved because of mass). Any paths which follow the local curvature of spacetime (free-fall) are inertial frames, and we know how to do physics in inertial frame so we need to find out what these paths are.

3.1 Parallel transport

Some strange things happen when space is curved rather than flat. In flat space, we can take a vector from point and it keeps its direction - this is called parallel transport. This is important as when we do differentials we're comparing a vector at some point with a vector at another point. In curved space e.g. the surface of a sphere it just doesn't happen and the direction the vector points at the end of the path depends on the PATH as well as the start and end points. This is a really big problem when we start to look at differentiation.

Lets think some more about this, and have our vector $\vec{\lambda} = \lambda^a(u)\vec{e_a}$ defined along some curve given by coordinates $x^a(u)$. Suppose we tried to define derivitive in the 'obvious' way as

$$\frac{d\lambda^a}{du} = \frac{\lim}{\delta u \to 0} \frac{[\lambda^a(u + \delta u) - \lambda^a(u)]}{\delta u}$$

For a start this isn't a tensor as it doesn't transform.

$$\frac{d\lambda^{a'}}{du} = \frac{d(\partial x^{a'}/\partial x^b \lambda^b)}{du} = \frac{\partial x^{a'}}{\partial x^b}\frac{d\lambda^b}{du} + \lambda^b \frac{d}{du}\frac{\partial x^{a'}}{\partial x^b} = \frac{\partial x^{a'}}{\partial x^b}\frac{d\lambda^b}{du} + \frac{\partial^2 x^{a'}}{\partial x^b \partial x^c}\frac{dx^c}{du}\lambda^b$$

Theres a funny extra term, which comes from the fact that $\lambda^a(u+\delta u)$ and $\lambda^a(u)$ are located at different points. In general the coordiate transforms depend on position so $\partial x^{a'}/\partial x^b$ evaluated at $u+\delta u$ is not the same as $\partial x^{a'}/\partial x^b$ at u. For differentiation to give a vector (tensor) we must take component differences at the same point. In flat space we slide one of the vectors to the other one by moving one of the vectors parallel to itself (called parallel transport).

So how do we parallelly transport a vector ? We don't change its length or direction so

$$\frac{d\vec{\lambda}}{du} = 0 = \frac{d\lambda^a \vec{e_a}}{du} = \frac{d\lambda^a}{du}\vec{e_a} + \lambda^a \frac{d\vec{e_a}}{du} = \frac{d\lambda^a}{du}\vec{e_a} + \lambda^a \frac{\partial \vec{e_a}}{\partial x^b}\frac{dx^b}{du}$$

There is this nasty component $\partial\vec{e_a}/\partial x^b$ but since this is the derivitive of a vector then it is itself a vector so we can write it as a linear combination of basis vectors

$$\frac{\partial \vec{e_a}}{\partial x^b} = \Gamma^c_{ab}\vec{e_c}$$

these are called Christoffel symbols or connection coefficients. We have written them in tensor notation but THEY DO NOT TRANSFORM AS TENSORS IN GENERAL.

$$= \frac{d\lambda^a}{du}\vec{e_a} + \lambda^a\Gamma^c_{ab}\frac{dx^b}{du}\vec{e_c} = \frac{d\lambda^a}{du}\vec{e_a} + \lambda^b\Gamma^i_{bc}\frac{dx^c}{du}\vec{e_a}$$

so since this is all the same vector component we can see that

$$\frac{d\lambda^a}{du} + \lambda^b\Gamma^a_{bc}\frac{dx^c}{du} = 0$$

So if we parallelly transport a vector λ at u to $u+\delta u$ we have

$$\frac{\delta\lambda^a}{\delta u} + \lambda^b\Gamma^a_{bc}\frac{\delta x^c}{\delta u} = 0$$

$$\delta\lambda^a = -\lambda^b\Gamma^a_{bc}\frac{\delta x^c}{\delta u}\delta u$$

$$\delta\lambda^a = -\lambda^b\Gamma^a_{bc}\delta x^c$$

so the parallely transported vector is $\hat{\lambda}^a(u+\delta u) = \lambda^a(u) - \lambda^b\Gamma^a_{bc}\delta x^c$, i.e. its PATH DEPENDANT in general. We can compare this with the vector

$\vec{\lambda}(u+\delta u)$ and use this to define absolute derivitive. The point is that $\vec{\lambda}(u+\delta u)$ contains two differences $\vec{\lambda}(u)$ - firstly there can be real physical differences - maybe this contravariant tensor is telling us about velocity and maybe there has been some real physics going on - accelerations and decelerations, that mean that $\vec{\lambda}(u+\delta u)$ is different from $\vec{\lambda}(u)$. BUT THERE CAN ALSO BE CHANGES BETWEEN THESE TWO VECTORS JUST BECAUSE THE SPACE IS CURVED. And we want to take out these 'space changes' which are what the parallelly transported vector tells us about, so that we can see the real physical changes. So these are

$$\frac{\lambda^a(u+\delta u)-\hat{\lambda^a}(u+\delta u)}{\delta u}=\frac{\lambda^a(u)+\frac{d\lambda^a}{du}\delta u-\lambda^a(u)+\lambda^b\Gamma^a_{bc}\delta x^c}{\delta u}=\frac{d\lambda^a}{du}+\lambda^b\Gamma^a_{bc}\frac{\delta x^c}{\delta u}$$

so

$$\frac{D\lambda^a}{du}=\frac{d\lambda^a}{du}+\lambda^b\Gamma^a_{bc}\frac{dx^c}{du}$$

We saw previously that $d\lambda^a/du$ didn't transform as a tensor - there were two bits to the coordinate transformation, one which was the standard tensor transformation but there was a second term as well which made it NOT a tensor. Well, it turns out that the way we defined christoffel symbols (which are NOT tensors) ALSO has such a second term but with the opposite sign which CANCELS out with the one in the $d\lambda^a/du$ so this ABSOLUTE derivative IS A TENSOR - a standard contravariant tensor - as it transforms as a tensor.

3.2 Covariant derivative

We can do this in terms of coordinates as opposed to a single parameter, and remove that path dependance. We had

$$\frac{D\lambda^a}{du}=\frac{d\lambda^a}{du}+\lambda^b\Gamma^a_{bc}\frac{dx^c}{du}=\frac{\partial\lambda^a}{\partial x^c}\frac{dx^c}{du}+\lambda^b\Gamma^a_{bc}\frac{dx^c}{du}=\left(\frac{\partial\lambda^a}{\partial x^c}+\lambda^b\Gamma^a_{bc}\right)\frac{dx^c}{du}$$

So covariant derivative, denoted $;c$, of a contravarient tensor is

$$\lambda^a_{;c}=\frac{\partial\lambda^a}{\partial x^c}+\Gamma^a_{bc}\lambda^b=\frac{\partial\lambda^a}{\partial x^c}+\Gamma^a_{bc}\lambda^b$$

This is obviously closely related to the absolute derivative, which is a (1,0) tensor, so this is a (1,1) tensor

3.3 Geodesic equations

So now we can get the geodesic paths. We could define a geodesic as the shortest distance between two points but actually its easier to start from another property related to parallel transport namely that geodesics are curves which parallel transport their tangent vectors. In effect this is saying that the geodesics have tangent vectors all pointing along tangents to the spatial curvature.

Suppose we take some path u and look its tangent vector $\vec{\tau}$ to any point on the curve $\vec{r}(u) = x^a(u)\vec{e_a}$.

$$\vec{\tau} = \frac{d\vec{r}}{du} = \frac{\partial \vec{r}}{\partial x^a}\frac{dx^a}{du} = \frac{dx^a}{du}\vec{e_a}$$

as basis vectors are defined as $\partial\vec{r}/\partial x^a = \vec{e_a}$. So the components $\tau^a = dx^a/du$. From the parallel transport equation above, parallel transport implies

$$\frac{d\tau^a}{du} + \tau^b\Gamma^a_{bc}\frac{dx^c}{du} = 0$$

substitute in$\tau^a = dx^a/du$ from the definition of tangent vector and this becomes

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

This works for any parameter linearly related to path length $t = A + Bu$ so the **affinely parameterized geodesic equation** in an N dimensional manifold is

$$\frac{d^2x^a}{dt^2} + \Gamma^a_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt} = 0$$

So now we know how to do geodesic paths, as well as derivatives. But they all contain these these connection coefficiants/christoffel symbols.

3.4 Christoffel symbols and the metric

What are these horrors ? well they come from the derivitives of the tangent basis vectors, but these themselves are defined as $\vec{e_a} = \partial\vec{r}/\partial x^a$ so

$$\Gamma^c_{ab}\vec{e_c} = \frac{\partial\vec{e_a}}{\partial x^b} = \frac{\partial^2\vec{r}}{\partial x^a\partial x^b} = \frac{\partial^2\vec{r}}{\partial x^b\partial x^a} = \frac{\partial\vec{e_b}}{\partial x^a} = \Gamma^c_{ba}\vec{e_c}$$

so they are symmetric (changing the order of integration assumes that the second partial derivitives are continuous). To get some more about them the next step is to look at how the metric changes as a function of the coordinates.

$$\frac{\partial g_{ab}}{\partial x^c} = \frac{\partial \vec{e_a}.\vec{e_b}}{\partial x^c} = \frac{\partial \vec{e_a}}{\partial x^c}.\vec{e_b} + \vec{e_a}.\frac{\partial \vec{e_b}}{\partial x^c} = \Gamma^d_{ac}\vec{e_d}.\vec{e_b} + \Gamma^f_{bc}\vec{e_a}.\vec{e_f} = \Gamma^d_{ac}g_{db} + \Gamma^f_{bc}g_{af}$$

relabel to get:

$$\frac{\partial g_{ab}}{\partial x^c} = \Gamma^d_{ac}g_{db} + \Gamma^d_{bc}g_{ad}$$

then move all suffixes around to get

$$\frac{\partial g_{bc}}{\partial x^a} = \Gamma^d_{ba}g_{dc} + \Gamma^d_{ca}g_{bd}$$

$$\frac{\partial g_{ca}}{\partial x^b} = \Gamma^d_{cb}g_{da} + \Gamma^d_{ab}g_{cd}$$

add the first two and subtract the third to get an expression for the Christoffel symbols in terms of the derivities of the metric - SO IF THE METRIC IS CONSTANT THEN THE CHRISTOFFEL SYMBOLS ARE ZERO. So

$$2\Gamma^d_{ca}g_{db} = \frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b}$$

so multiply by $\frac{1}{2}g^{fb}$ to get

$$\Gamma^f_{ca} = \frac{1}{2}g^{fb}(\frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b})$$

So we can calculate these horrors!! and hence we know how to calculate geodesic paths (ie we can find inertial frames!). And in passing we learnt how to do derivatives in curved space as well. Not bad for 1 lecture!

3.5 Example: flat space

The metric for flat space DOES NOT DEPEND ON POSITION. So whatever coordinates we use (cartesian or spherical polar) gives that the partial derivatives of the metric are ZERO. so $\Gamma^a_{bc} = 0$. so the geodesics are

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

$$\frac{d^2x^a}{du^2} = 0$$

$$\frac{dx^a}{du} = \text{constant} = A$$

$$x^a = Au + B$$

ie the equation of a straight line! this is correct! geodesics, ie inertial frames in flat space are Newtonian (if there is no net force on an object then it willl remain at rest or move with constant velocity in a straight line). Then parallel transport of vectors doesn't do anything to them!

$$\frac{d\lambda^a}{du} + \lambda^b \Gamma^a_{bc} \frac{dx^c}{du} = 0$$

$$\frac{d\lambda^a}{du} = 0$$

And absolute derivatives are the same as 'normal' derivatives $D\lambda^a/du = d\lambda^a/du + \lambda^b \Gamma^a_{bc} dx^c/du = d\lambda^a/du$

3.6 Example: surface of a sphere

We had previously worked out the basis vectors for the surface of a sphere and the metric (section 2.10) so I'll just repeat them here, working in coordinates where $x^1 = \theta$, $x^2 = \phi$. Then $\vec{e_1} = a\cos x^1 \cos x^2 \mathbf{i} + a\cos x^1 \sin x^2 \mathbf{j} - a \sin x^1 \mathbf{k}$ and $\vec{e_2} = -a \sin x^1 \sin x^2 \mathbf{i} + a \sin x^1 \cos x^2 \mathbf{j}$, giving $ds^2 = a^2(dx^1)^2 + a^2 \sin^2 x^2 (dx^2)^2 = a^2 d\theta^2 + a^2 \sin^2 \theta d\phi^2$ i.e. $g_{AB} = \text{diag}(a^2, a^2 \sin^2 x^1)$. so the metric IS a function of coordinates - its got an $x^1 = \theta$ term but no $x^2 = \phi$ term. So all the $\partial_C g_{AB} = 0$ for $c = 2$ ie $\partial_\phi g_{AB} = 0$, but all the $\partial_1 g_{AB}$ need to be calculated. But the metric is diagonal, so all the off diagonal terms are zero so $\partial_1 g_{AB} = 0$ unless A=B.

$$\partial_1 g_{11} = \frac{\partial a^2}{\partial x^1} = 0$$

$$\partial_1 g_{22} = \frac{\partial a^2 \sin^2 x^1}{\partial x^1} = 2a^2 \sin x^1 \cos x^1$$

Christoffel symbols are defined as

$$\Gamma^f_{ca} = \frac{1}{2} g^{fb} (\partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca})$$

so we need to also find the covariant metric components g^{AB} from $g^{AB}g_{BC} = \delta^A_C$ so $g^{AB} = \mathrm{diag}[1/a^2, 1/(a^2\sin^2 x^1)]$

So in our 2D space we have

$$\Gamma^1_{11} = 1/2g^{1B}(\partial_1 g_{1B} + \partial_1 g_{B1} - \partial_B g_{11})$$

as $g^{1B} = 0$ except for $B = 1$ then this collapses to

$$\Gamma^1_{11} = 1/2g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = 1/2\ 1/a^2 0 = 0$$

$$\Gamma^1_{12} = \Gamma^1_{21} = 1/2g^{1B}(\partial_2 g_{1B} + \partial_1 g_{B2} - \partial_B g_{12})$$

again g^{1B} is only nonzero for $B = 1$ since the metric is diagonal so

$$\Gamma^1_{12} = \Gamma^1_{21} = 1/2g^{11}(\partial_1 g_{11} + \partial_2 g_{11} - \partial_1 g_{12}) = 1/2\ 1/a^2 0 = 0$$

$$\Gamma^1_{22} = 1/2g^{1B}(\partial_2 g_{2B} + \partial_2 g_{B2} - \partial_B g_{22}) = 1/2\ 1/a^2(0 + 0 - \partial_1 g_{22})$$

$$= 1/21/a^2(-2a^2\sin x^1\cos x^1 = -\sin x^1\cos x^1$$

$$\Gamma^2_{11} = 1/2g^{2B}(\partial_1 g_{1B} + \partial_1 g_{B1} - \partial_B g_{11}) = 1/2g^{22}(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}) = 0$$

$$\Gamma^2_{12} = \Gamma^2_{21} = 1/2g^{2B}(\partial_2 g_{1B} + \partial_1 g_{B2} - \partial_B g_{12}) = 1/2g^{22}(\partial_2 g_{12} + \partial_1 g_{22} - \partial_2 g_{12})$$

$$= 1/2\ 1/(a^2\sin^2 x^1)\partial_1 g_{22} = 1/2\ 1/(a^2\sin^2 x^1)2a^2\sin x^1\cos x^1 = \cot x^1$$

$$\Gamma^2_{22} = 1/2g^{2B}(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) = 0$$

Now do geodesics knowing that the only non-zero christoffel symbols are $\Gamma^1_{22} = -\sin x^1\cos x^1$, and $\Gamma^2_{12} = \Gamma^2_{21} = \cot x^1$. Geodesic paths satisfy the equation

$$\frac{d^2x^A}{du^2} + \Gamma^A_{BC}\frac{dx^B}{du}\frac{dx^C}{du} = 0$$

suppose the path u is just a change in x^1 then $u = ax^1$ so there is no dependance on x^2 i.e. x^2 =constant so $dx^2/du = 0$. While for x^1 we have $dx^1/du = 1/a$ and $d^2x^1/du^2 = 0$. For x^1 then

$$\frac{d^2x^1}{du^2} + \Gamma^1_{BC}\frac{dx^B}{du}\frac{dx^C}{du} = \frac{d^2x^1}{du^2} + \Gamma^1_{22}\frac{dx^2}{du}\frac{dx^2}{du} = 0$$

so this looks good. Check also for x^2

$$\frac{d^2x^2}{du^2} + \Gamma^2_{BC}\frac{dx^B}{du}\frac{dx^C}{du} = \frac{d^2x^2}{du^2} + \Gamma^2_{12}\frac{dx^1}{du}\frac{dx^2}{du} + \Gamma^2_{21}\frac{dx^2}{du}\frac{dx^1}{du} = 0$$

so all such paths are geodesics. - these are GREAT CIRCLES.

3.7 Easier Christoffel Symbols for geodesic equations

But the Christoffel symbols we defined as derivites of the metric and it all looked horrid to calculate. And geodesics in generality are horrid to calculate as well. An easier way (note I didn't say EASY) is to use the Euler–Lagrange equations. We could have solved for the geodesics (at least for matter particles) by saying that geodesics are paths which give the shortest distance between two points, i.e. we are looking for the extremal path which has

$$\delta \int ds = 0$$

But this is often what we do in classical mechanics, where we look for the minimum energy path by getting the Lagrangian $L = T - V$ as the sum of kinetic T and potential V energies, and then finding the minimum energy path by integrating this over time i.e. $\delta \int L dt = 0$. For geodesics then there are no potential energy terms so $V = 0$ so we are looking only at kinetic energy, so $\delta \int T dt = 0$.

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We'll do it in terms of per unit mass so $T = 1/2v^2 = 1/2[(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2]$, or in metric terms (Euclidean 3D flat space)

$$T = \frac{1}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j = \frac{1}{2} \Big(\frac{ds}{dt} \Big)^2$$

So the minimum energy condition (Hamiltons principle in classical mechanics) gives

$$\delta \int T dt = \delta \int \Big(\frac{ds}{dt}\Big)^2 dt = 0$$

i.e. this is basically the same as the minimum path requirement which defines our geodesic which is $\delta \int (ds/dt)dt = 0$.

We know that in classical mechamics that the solution with the minimum energy satisfies the Euler-Lagrange equations i.e.

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}^i}\Big) - \frac{\partial L}{\partial x^i} = 0$$

where $L = 1/2 g_{ij}\dot{x}^i\dot{x}^j$. If we wanted we could do long and tedious algebra on these equations to get the equation for a geodesic path, and it would turn out to be of the same form as we got before with far less work by doing parallel transport of a vector i.e.

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$$

or, writing it explicitly

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

While there are obviously some technical problems in carrying this over into our 4D spacetime, where the metric is indefinite (can be +ve, 0 or -ve), if you do it, then it comes out to be the same as above except we are now going to take our derivative wrt some affine parameter u related to the invariant path length rather than to the COORDINATE t. So we could write the geodesic equation as before but where the dot now stands for derivative wrt the affine parameter u. So the geodesic paths are $\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$.

But then the midstep Euler–Lagrange equations still hold, so we have the condition that

$$\frac{d}{du}\Big(\frac{\partial L}{\partial \dot{x}^\alpha}\Big) - \frac{\partial L}{\partial x^\alpha} = 0$$

And these equations give us directly the Christoffel symbols by comparison with the geodesic equations. It is exactly the same as if you have two equations for the same thing, one which says $Ax + B = 0$ and another which says $f(x)x + g(x) = 0$ where $f(x)$ and $g(x)$ are horrendous functions, then we can either get f and g by calculating the horrendous functions or we can just say by comparison that $f(x) = A$ and $g(x) = B$.

3.8 Example using the Robertson Walker metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - [R(t)]^2[(1-kr^2)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2]$$

where k is a constant, and $c = 1$. So the Lagrangian of the system is

$$L(\dot{x}^\sigma, x^\sigma) = 1/2[\dot{t}^2 - [R(t)]^2((1-kr^2)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)]$$

Lets just look at the time component. Partial differntialtion gives

$$\frac{\partial L}{\partial \dot{t}} = \dot{t}$$

$$\frac{\partial L}{\partial t} = -R\frac{dR}{dt}[(1-kr^2)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2]$$

So the Euler-Lagrange equation gives us the condition

$$\frac{d}{du}\Big(\frac{\partial L}{\partial \dot{t}}\Big) - \frac{\partial L}{\partial t} = 0$$

$$\frac{d\dot{t}}{du} + R\frac{dR}{dt}[(1-kr^2)^{-1}\dot{r}^2 + r^2\dot{\theta} + r^2\sin^2\theta\dot{\phi}] = 0$$

$$\ddot{t} + R\frac{dR}{dt}[(1-kr^2)^{-1}\dot{r}^2 + r^2\dot{\theta} + r^2\sin^2\theta\dot{\phi}] = 0$$

But the geodesic equation is $\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$. so for parameter t i.e. $\alpha = 0$ we have

$$\ddot{t} + \Gamma^0_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$$

so we compare equations are read them straight off.

$$\Gamma^0_{11} = R\frac{dR}{dt}(1-kr^2)^{-1}, \quad \Gamma^0_{22} = R\frac{dR}{dt}r^2, \quad \Gamma^0_{33} = R\frac{dR}{dt}r^2\sin^2\theta$$

and all the rest of the $\Gamma^0_{\beta\gamma}$ are zero.

3.9 Geodesics in weakly curved space = Newtonian gravity!

Now lets go apply some of some of this new found knowledge of christoffel symbols to REAL PHYSICS! Suppose we are looking at a material particle in free fall. It goes along a geodesic so

$$\frac{d^2x^\alpha}{du^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{du}\frac{dx^\gamma}{du} = 0$$

where u is an affine parameter, linearly related to path length s. well, proper time τ is an affine parameter as $c^2 d\tau^2 = ds^2$ so

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

Go to the Newtonian limit in a stationary gravitational field, and $v \ll c$.

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$$

Left hand side is acceleration - so the right hand side must be gravity! $dx^0/d\tau = cdt/d\tau \sim c$ while $dx^i/d\tau = v^i dt/d\tau \sim v \ll c$ so we can neglect the spatial derivatives in comparison to the time ones.

$$\frac{d^2x^\alpha}{d\tau^2} \approx -\Gamma^\alpha_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = -c^2\Gamma^\alpha_{00}\Big(\frac{dt}{d\tau}\Big)^2 \approx -c^2\Gamma^\alpha_{00}$$

The field is stationary (in time) so $\partial g_{\mu\nu}/\partial x^0 = 0$ i.e. only the spatial derivatives of the metric can matter.

$$\Gamma^f_{ca} = \frac{1}{2}g^{fb}(\partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca})$$

$$\Gamma^\alpha_{00} = \frac{1}{2}g^{\alpha\beta}(\partial_0 g_{0\beta} + \partial_0 g_{\beta 0} - \partial_\beta g_{00}) = -\frac{1}{2}g^{\alpha\beta}\partial_\beta g_{00}$$

If the field is weak then we can write the metric as s sum of the flat spacetime metric which we'll now call $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$ plus some little perturbation: $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ so

$$\Gamma^\alpha_{00} = -\frac{1}{2}g^{\alpha\beta}\partial_\beta g_{00} = -\frac{1}{2}\eta^{ij}\partial_j h_{00}$$

where the last step (going to 3D rather than 4D) follows as the time derivative of the metic is zero for a static field. The geodesic (equation of motion) becomes

$$\frac{d^2x^\alpha}{d\tau^2} \approx \frac{d^2x^\alpha}{dt^2} \approx \frac{c^2}{2}\eta^{ij}\partial_j h_{00} = \nabla(-c^2h_{00}/2)$$

as $\eta^{ij} = -\delta^i_j$. But we know that if we are a distance r from a star of mass M then we measure an acceleration (per unit mass) of

$$\frac{d^2x^i}{dt} = \nabla\phi \qquad \phi = -GM/r$$

so

$$h_{00} = \frac{2\phi}{c^2} + \text{const} = \frac{2\phi}{c^2}$$

as for $r \to \infty$ then $\phi \to 0$ and $g_{\alpha\beta} \to \eta_{\alpha\beta}$ so $h_{00} \to 0$. So this gives us

$$g_{00} = \eta_{00} + h_{00} = 1 + \frac{2\phi}{c^2}$$

So we started from our geodesic equation which only told us about curved space. But we ended up with something that looks exactly like gravity!

3.10 Summary: Tensor derivatives

Absolute derivative of a contravariant tensor over some path

$$\frac{D\lambda^a}{du} = \frac{d\lambda^a}{du} + \lambda^b\Gamma^a_{bc}\frac{dx^c}{du}$$

gives a tensor field of same type (contravariant first order) in this case. The first bit takes into account both real physical changes in λ^a AND the way the curvature of the space can swing the vector while the second one takes out the swinging the vector from the curvature of space alone. So if there is no physical change then $D\lambda^a/du = 0$ and the vector is simply parallelly transported. Absolute derivative obeys all the normal rules for derivatives so $D(\lambda^a + k\mu^a)/du = D(\lambda^a)/du + kD(\mu^a)/du$ (linear) and $D(\lambda^a\mu^a)/du = \mu^a D(\lambda^a)/du + \lambda^a D(\mu^a)/du$ (Leibniz' rule).

Absolute derivative of a scalar

There is no space swing with a simple number, ϕ. so $D\phi/du = d\phi/du$

Absolute derivative of covariant tensors

$D\phi/du = d\phi/du$ so let $\phi = \lambda^a \mu_a$ then

$$\frac{D(\lambda^a \mu_a)}{du} = \frac{d(\lambda^a \mu_a)}{du}$$

$$\mu_a \frac{D\lambda^a}{du} + \lambda^a \frac{D\mu_a}{du} = \mu_a \frac{d\lambda^a}{du} + \lambda^a \frac{d\mu_a}{du}$$

$$\mu_a(\frac{d\lambda^a}{du} + \Gamma^a_{bc}\lambda^b \frac{dx^c}{du}) + \lambda^a \frac{D\mu_a}{du} = \mu_a \frac{d\lambda^a}{du} + \lambda^a \frac{d\mu_a}{du}$$

$$\lambda^a \frac{D\mu_a}{du} = \mu_a \frac{d\lambda^a}{du} + \lambda^a \frac{d\mu_a}{du} - \mu_a(\frac{d\lambda^a}{du} + \Gamma^a_{bc}\lambda^b \frac{dx^c}{du})$$

$$\lambda^a \frac{D\mu_a}{du} = \lambda^a \frac{d\mu_a}{du} - \mu_a \Gamma^a_{bc}\lambda^b \frac{dx^c}{du} = \lambda^a(\frac{d\mu_a}{du} - \Gamma^b_{ac}\mu_b \frac{dx^c}{du})$$

$$\frac{D\mu_a}{du} = \frac{d\mu_a}{du} - \Gamma^b_{ac}\mu_b \frac{dx^c}{du}$$

Absolute derivative of higher order tensors

For example, to get the absolute derivative of a mixed tensor τ^a_b then look at the special case where $\tau^a_b = \lambda^a \mu_b$

$$\frac{D\tau^a_b}{du} = \lambda^a \frac{D\mu_b}{du} + \mu_b \frac{D\lambda^a}{du} = \lambda^a(\frac{d\mu_b}{du} - \Gamma^d_{bc}\mu_d \frac{dx^c}{du}) + \mu_b(\frac{d\lambda^a}{du} + \Gamma^a_{dc}\lambda^d \frac{dx^c}{du})$$

$$= \frac{d(\lambda^a \mu_b)}{du} + \mu_b \Gamma^a_{dc}\lambda^d \frac{dx^c}{du} - \lambda^a \Gamma^d_{bc}\mu_d \frac{dx^c}{du} = \frac{d(\tau^a_b)}{du} + \Gamma^a_{dc}\tau^d_b - \Gamma^d_{bc}\tau^a_d \frac{dx^c}{du}$$

Covariant derivative

$$\lambda^a{}_{;}c = \frac{\partial \lambda^a}{\partial x^c} + \Gamma^a_{bc}\lambda^b$$

rules are as above, so for a second order covariant tensor we have

$$\lambda_{ab;c} = \frac{\partial \lambda_{ab}}{\partial x^c} - \Gamma^d_{ac}\lambda_{db} - \Gamma^d_{bc}\lambda_{ad}$$

(in homework set 3 I'll get you to derive all of these). Covariant differention forms a tensor field of 1 higher covariant order than the original tensor field.

Covariant derivative of the metric

In getting the Christoffel symbols (section 3.4) in terms of the metric we had

$$\frac{\partial g_{ab}}{\partial x^c} = \frac{\partial \vec{e_a}.\vec{e_b}}{\partial x^c} = \frac{\partial \vec{e_a}}{\partial x^c}.\vec{e_b} + \vec{e_a}.\frac{\partial \vec{e_b}}{\partial x^c}$$

$$= \Gamma^d_{ac}\vec{e_d}.\vec{e_b} + \vec{e_a}.\vec{e_d}\Gamma^d_{bc} = \Gamma^d_{ac}g_{db} + \Gamma^d_{bc}g_{ad}$$

rearrange this to get

$$\frac{\partial g_{ab}}{\partial x^c} - \Gamma^d_{ac}g_{db} - \Gamma^d_{bc}g_{ad} = 0$$

but this is the definition of the covariant derivative of a second order covarient tensor. so $g_{ab;c} = 0$, and we can similarly get that $g^{ab}_{;c} = 0$. This means that index raising and lowering can be swapped in and out of covariant differention. $R^a = g^{ab}R_b$. Then $R^a_{;c} = (g^{ab}R_b)_{;c} = g^{ab}_{;c}R_b + g^{ab}R_{b;c} = g^{ab}R_{b;c}$

4 Tensor and Physical Curvature

So we've seen that gravity=curvature. So it would be nice to have a way to quantify curvature. There is one (but it isn't nice!).

4.1 Riemann Curvature Tensor

Key thing about curvature is that neighbouring geodesics get further apart (or closer together) at a rate depending on the local curvature. Flat space, no forces, geodesics are straight lines. The distance between two geodesics then increases/decreases linearly as a function of path length along the geodesic. So the second derivative of the distance between them is zero. For curved space then the geodesics don't separate linearly - the second derivative will not be zero, and this gives us a way to quantify the curvature of spacetime.

Take two geodesics γ and $\tilde{\gamma}$ with coodinates $x^a(u)$ and $\tilde{x}^a(u)$. The distance between them is $\zeta^a(u) = \tilde{x}^a(u) - x^a(u) =$. These are geodesics so

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

$$\frac{d^2\tilde{x}^a}{du^2} + \tilde{\Gamma}^a_{bc}\frac{d\tilde{x}^b}{du}\frac{d\tilde{x}^c}{du} = 0$$

but

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \frac{\partial \Gamma^a_{bc}}{\partial x^d}(\tilde{x}^d - x^d) = \Gamma^a_{bc} + \partial_d \Gamma^a_{bc}\zeta^d$$

while subtracting the geodesics gives

$$\frac{d^2\tilde{x}^a}{du^2} + \tilde{\Gamma}^a_{bc}\frac{d\tilde{x}^b}{du}\frac{d\tilde{x}^c}{du} - \frac{d^2 x^a}{du^2} - \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

$$\frac{d^2(x^a+\zeta^a)}{du^2} + (\Gamma^a_{bc} + \partial_d\Gamma^a_{bc}\zeta^d)\frac{d(x^b+\zeta^b)}{du}\frac{d(x^c+\zeta^c)}{du} - \frac{d^2x^a}{du^2} - \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

Let dot denote derivative with respect to path length, so this becomes

$$\ddot{x}^a + \ddot{\zeta}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a_{bc}\dot{x}^c\dot{\zeta}^b + \Gamma^a_{bc}\dot{\zeta}^b\dot{\zeta}^c +$$

$$\partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{\zeta}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^c\dot{\zeta}^b + \partial_d\Gamma^a_{bc}\zeta^d\dot{\zeta}^b\dot{\zeta}^c - \ddot{x}^a - \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$$

keep only first order terms in ζ and $\dot{\zeta}$ so ignore $\zeta^a\zeta^b$, $\zeta^a\dot{\zeta}^b$ and $\dot{\zeta}^a\dot{\zeta}^b$ - this is equivalent to saying the geodesics are originally close together. NB we keep the second derivative because if we had $y = x^2$ then $\dot{y} = 2x$ and $\ddot{y} = 2$. so if x is small then x and $\dot{x}$ are small but there is nothing you can do about 2!! so we keep second derivatives.

$$\ddot{\zeta}^a + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

We have this term $\ddot{\zeta}^a$. We are looking for a tensor equation, and we'd get something with this term that transformed as a tensor if we had $D^2\zeta^a/du^2$.

$$\frac{D^2\zeta^a}{du^2} = \frac{D}{du}(\dot{\zeta}^a + \Gamma^a_{bc}\zeta^b\dot{x}^c) = \frac{D}{du}\dot{\zeta}^a + \frac{D}{du}(\Gamma^a_{bc}\zeta^b\dot{x}^c)$$

$$= \ddot{\zeta}^a + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \frac{d}{du}(\Gamma^a_{bc}\zeta^b\dot{x}^c) + \Gamma^a_{ef}(\Gamma^e_{bc}\zeta^b\dot{x}^c)\dot{x}^f$$

$$= \ddot{\zeta}^a + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \frac{d\Gamma^a_{bc}}{du}\zeta^b\dot{x}^c + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \Gamma^a_{bc}\zeta^b\ddot{x}^c + \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f$$

We can substitute for $\ddot{x}$ from the geodesic equation along γ, and take the christoffel symbol derivative with respect to coordinates rather than the affine parameter

$$= \ddot{\zeta}^a + 2\Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^b\dot{x}^c\dot{x}^d - \Gamma^a_{bc}\zeta^b\Gamma^c_{ef}\dot{x}^e\dot{x}^f + \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f$$

so we then have an expression for $\ddot{\zeta}$ as

$$\ddot{\zeta}^a = \frac{D^2\zeta^a}{du^2} - 2\Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c - \partial_d\Gamma^a_{bc}\zeta^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\zeta^b\Gamma^c_{ef}\dot{x}^e\dot{x}^f - \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f$$

Substitute for $\ddot{\zeta}^a$ into our original equation for the separation of two geodesics

$$\ddot{\zeta}^a + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

$$\frac{D^2\zeta^a}{du^2} - 2\Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c - \partial_d\Gamma^a_{bc}\zeta^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\zeta^b\Gamma^c_{ef}\dot{x}^e\dot{x}^f - \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c$$
$$+\Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

$$\frac{D^2\zeta^a}{du^2} - \Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c - \partial_d\Gamma^a_{bc}\zeta^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\zeta^b\Gamma^c_{ef}\dot{x}^e\dot{x}^f - \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f + \Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

But the Christoffel symbols are symmetric so $\Gamma^a_{bc}\dot{\zeta}^b\dot{x}^c = \Gamma^a_{cb}\dot{\zeta}^c\dot{x}^b = \Gamma^a_{bc}\dot{\zeta}^c\dot{x}^b$

$$\frac{D^2\zeta^a}{du^2} - \partial_d\Gamma^a_{bc}\zeta^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\zeta^b\Gamma^c_{ef}\dot{x}^e\dot{x}^f - \Gamma^a_{ef}\Gamma^e_{bc}\zeta^b\dot{x}^c\dot{x}^f + \partial_d\Gamma^a_{bc}\zeta^d\dot{x}^b\dot{x}^c = 0$$

swap indices around and get

$$\frac{D^2\zeta^a}{du^2} + (\Gamma^a_{be}\Gamma^e_{cd} - \partial_d\Gamma^a_{bc} - \Gamma^a_{ed}\Gamma^e_{bc} + \partial_b\Gamma^a_{dc})\zeta^b\dot{x}^c\dot{x}^d = 0$$

So here we have something which should equal zero if the space is flat, and not zero if the space is curved. This can all be stuck together into the Riemann curvature tensor

$$\frac{D^2\zeta^a}{du^2} + R^a_{cbd}\zeta^b\dot{x}^c\dot{x}^d = 0$$

where

$$R^a_{cbd} = (\Gamma^a_{be}\Gamma^e_{cd} - \partial_d\Gamma^a_{bc} - \Gamma^a_{ed}\Gamma^e_{bc} + \partial_b\Gamma^a_{dc})$$

Geodesics in flat space maintain their separation. Those in curved space do not.

4.2 Newtonian gravity meaning of curvature tensor

In Newtonian gravity then two particles initially at rest at some distance R from a star of mass M are accelerated radially by the gravitational force GM/R^2. Since they are in free fall they are travelling on geodesics, and these geodesics are getting closer together as the particles fall towards the star. Resolving the radial force on one of the stars into a force along the same direction as the other radial force, and one perperdicular to it then there is a force towards the other particle of size $GM/R^2\zeta/R$ where ζ is the separation between the two particles. Thus there is an acceleration term between the two particles of magnitude which is pulling them together so

$$\frac{d^2\zeta}{dt^2} = -\frac{GM}{R^3}\zeta$$

Convert this now to path length using dirty astrophysicists approximations so $ds^2/dt^2 \sim c^2$

$$\frac{d^2\zeta}{ds^2}\Big(\frac{ds}{dt}\Big)^2 = \frac{d^2\zeta}{ds^2}c^2 = -\frac{GM}{R^3}\zeta$$
$$\frac{d^2\zeta}{ds^2} = -\frac{GM}{c^2R^3}\zeta$$

Compare this with the curvature tensor

$$\frac{D^2\zeta^a}{du^2} = -R^a_{cbd}\zeta^b\dot{x}^c\dot{x}^d$$

Again we see explicitally that the tidal forces which we'd associate with gravity can be made simply from curved spacetime. The curvature tensor in some sense tells us about TIDAL forces - how gravity changes from point to point.

4.3 Stress-Energy Tensor

So the curvature of space is related to tidal forces, ie to M/R^3 which is the density of mass. We want to get some tensor equation (valid in any frame) such that curvature=gravity, so we want some tensor equation linking the Riemann curvature tensor and mass density. But it can't just be mass density. Einstein said that mass and energy are equivalent so in fact ALL forms of energy (not just mass) should gravitate. And in fact if this were so it would make some sense of SR. If you accelerate a particle, giving it more and more energy, then as you approach c then the mass increases rather than the velocity. Which is all a bit arbitrary unless all forms of energy gravitate because then as we add energy we increase the inertial (i.e. gravitational) mass. So what we want is a tensor equation linking energy density with curvature.

First we will see how to characterise energy density in SR. In SR we generally dealt with individual particles. where $E = \gamma m_o c^2$ which is the particle energy, $u^\mu = dx^\mu/d\tau$ is world velocity, $v^\mu = dx^\mu/dt$ is coordinate velocity and $p^\mu = m_0 u^\mu$ is the 4 momentum of the particle. A stationary particle has $p^\mu = m_0(c, \vec{0})$ while a moving one has $p^\mu = m_0\gamma(c, \vec{v}) = (E/c, \vec{p})$. We want to generalise these to look at density. This is a problem because of course density is a frame dependent quantity. The total number of particles in a given volume must be invariant but the volume changes by length contraction along the direction of motion, so if n_0 is the number density in the rest frame then $n_0\gamma$ is the number density as measured in a moving frame. But if we are now moving, then there is a flux of particles across the surfaces. i.e. we take two separate concepts density and flux, and make them into a single 4-vector the number-flux 4-vector $\tilde{N} = n_0\tilde{u}$ so this has components $N^\mu = n_0 u^\mu$. This is a 4-vector as it transforms in the right way for a contravariant tensor. In the rest frame then $N^\mu = n_0 dx^\mu/d\tau$. In a moving frame then $N^{\nu'} = n dx^{\nu'}/d\tau = n_0 dx^{\nu'}/dx^\mu dx^\mu/d\tau = dx^{\nu'}/dx^\mu N^\mu$. Alternatively we could just have said that we know $u^\mu = dx^\mu/d\tau$ is a 4 vector and so since N^μ is related to it only by invariant quantities (rest frame density) then this must also be a 4-vector.

So that is density done. But we wanted energy density. Lets stick in the rest frame for the time being, then if the particles have no velocity relative to each other (this is called **dust**) then the energy density is simply $m_0 c^2 n_0 = \rho_0 c^2$ where ρ_0 is the mass density in the rest frame. But if we then go to another frame we have to transform the energy AND the density - so it transforms by two factors of γ. So this needs to be a second order contravariant tensor. How about $T^{\mu\nu} = N^\mu p^\nu$ - in the rest frame then this would have only one component $T^{00} = n_0 c m_0 c = \rho_0 c^2$. That sounds like a good start. So $T^{\mu\nu} = N^\mu p^\nu = n u^\mu m_0 u^\nu = \rho_0 u^\mu u^\nu$.

But in general a gas HAS internal motion - particles are moving with respect to each other so there is no rest frame as such for each particle, only a rest frame for the gas as a whole and the **dust** approximation doesn't work.

We can instead define a general $T^{\mu\nu}$ which is the flow of the μ component of 4-momentum across a surface of constant ν. Then in the rest frame we have T^{00} as flux of 0th component of 4 momentum (E/c) across a surface of constant ct - energy density, cT^{0i} is energy flow per unit area along the i direction - this would be heat conduction. $T^{i0}c$ is the density of the i component of momentum. T^{ii} is the flow of the ith momentum component per unit area in the i direction - i.e. pressure across the i plane. T^{ij} is the flow of the ith momentum component across the j plane - these are viscous drags.

In the rest frame the viscous drags have to be equal (or the fluid as a whole rotates and its not a rest frame) so $T^{ij} = T^{ji}$. And if heat flows along a direction i then that energy will carry momentum and $T^{0i} = T^{i0}$ So the general stress-energy tensor is symmetric in the rest frame - so it must be **symmetric** everywhere.

A **perfect fluid** is one in which there is no heat conduction or viscous stresses. Then in the rest frame

$$T^{\mu\nu} = \mathrm{diag}(\rho_0 c^2, p, p, p)$$

This makes a lot of sense - for a relativistic gas the mean energy per particle is 3kT, so the total rest frame energy density $\rho_0 c^2 + n_0 3kT = \rho_0 c^2 + 3p$. We can already see something really fun. PRESSURE contributes to energy density, and energy density curves space. So for the interior of a star, as the fuel runs out then the star contracts and gravity gets stronger. so we require a larger pressure to hold the star up. But this presssure adds to the energy density, so increases gravity, so more pressure is needed so .... black holes!

Lets write the stress energy tensor for a perfect fluid as the sum of tensors

$$T^{\mu\nu} = (\rho_0 + p/c^2)u^\mu u^\nu - p\eta^{\mu\nu}$$

so we can see its a tensor as its made out of things that are either invariant (scalars) or tensors.

4.4 Conservation laws

The stress energy tensor for a perfect fluid in special relativity.

$$T^{\mu\nu} = (\rho_0 + p/c^2)u^\mu u^\nu - p\eta^{\mu\nu}$$

is a really compact way to count up all the contributions to the energy density, and its a tensor so it works in all (inertial) frames.

But the stress energy tensor also embodies more information. Energy is conserved. Rate of change of energy is $l^3\partial T^{00}/\partial t$. But this change is produced by net energy inflow/outflow along the 6 faces of the cube. The energy flow in is $l^2cT^{01}(x)$ and then energy flow out is $l^2cT^{01}(x+l)$ so net inward flow is $-l^3c\partial T^{01}/\partial x$. There are simialr contributions from the other 2 pairs of faces so we have

$$l^3\frac{\partial T^{00}}{\partial t} = -l^3c\frac{\partial T^{01}}{\partial x} - l^3c\frac{\partial T^{02}}{\partial y} - l^3c\frac{\partial T^{03}}{\partial z}$$

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i} = 0$$

$$\partial_\alpha T^{0\alpha} = 0$$

Momentum is also conserved. and that leads to

$$\partial_\alpha T^{i\alpha} = 0$$

In other words, conservation of both momentum and energy are embodied in

$$\partial_\alpha T^{\beta\alpha} = 0$$

as the stress-energy tensor is symmetric then there is only one divergance $\partial_\alpha T^{\beta\alpha} = \partial_\alpha T^{\alpha\beta}$.

4.5 strong equivalence principle - how to go from SR to GR

Any physical law which can be expressed in tensor notation in SR has EXACTLY the same form in a locally inertial frame of a curve spacetime (GR)! Locally inertial frames (free fall frames) are equivalent to saying we choose to look at a small piece of spacetime, where we can approximate any curved space to a locally flat space. In this choice of coordinates (local cartesian coordinates) then all the christoffel symbols are zero so standard diferentiation and covariant differentiation are the same. BUT once we go outside of this the curvature becomes apparent. So to make our tensor equations work in any general frame we need to swap standard differentiation (which is NOT a real tensor) $(\partial/\partial x^c = \partial_c =_{,c}$ with covariant differentiation $_{;c}$ (which IS a tensor). So this is sometimes called the comma goes to semicolon rule!

So if we wanted to generalise $T^{\mu\nu}$ and its conservation laws to GR we'd swap the metric tensor from flat space to generalised curved space so for a perfect fluid

$$T^{\mu\nu} = (\rho_0 + p/c^2)u^\mu u^\nu - pg^{\mu\nu}$$

and take the covariant deriavative rather than the partial derivative

$$T^{\beta\alpha}_{;\alpha} = 0$$

4.6 Tensors associated with the Riemann curvature tensor

Well, we have a bit of a problem. We want a tensor equation which has curvature=energy density. The stress energy tensor is second order, symmetric, with covariant derivative of zero. So we are looking for a way to denote curvature which is also a second order symmetric tensor with covariant derivative of zero. But the Riemann curvature tensor which completely embodies all the information about the curvature of space is a 4th order tensor. There is no way that a second order tensor can be equal to a 4th order one. So we need to CONTRACT the Riemann curvature tensor, preferably without losing any information about the spatial curvature!!!

If we were to contract R^a_{bcd} we could sum over one of the covariant indices with the contravariant one. But which covariant index - in principle $R^a_{acd} \neq R^a_{bad} \neq R^a_{bca}$. However, the Riemann curvature tensor has a large number of symmetries so instead of having N^4 components it actually has *only*(!) $N^2(N^2-1)/12$. These are tedious to derive so I've put that on a handout. The results are

Cyclic Identity

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0$$

Bianchi identity

$$R^a_{bcd;e} + R^a_{bde;c} + R^a_{bec;d} = 0$$

Some of the other quantities are best expressed in the fully covariant form of the Riemann curvature tensor

$$R_{abcd} = g_{ae} R^e_{bcd}$$

$$R_{abcd} = -R_{bacd}..........1$$

$$R_{abcd} = -R_{abdc}..........2$$

$$R_{abcd} = R_{cdab}...........3$$

so because of all these symmetries then we can contract the Riemann curvature tensor to a second order tensor and not lose TOO much information. We can contract it in 3 different ways R^a_{bca} or R^a_{bac} or R^a_{abc}. In general these would all be DIFFERNT tensors. But because of the symmetries these are not! From (1) $R_{abcd} = -R_{bacd}$ so if we sum over the first two indices then $R_{aacd} = -R_{aacd}$ so this can only be satsfied if $R_{aacd} = 0$ ie $R^a_{abc} = 0$. Also from (2) $R_{abcd} = -R_{abdc}$ so $R_{abca} = -R_{abac}$ so $R^a_{bca} = -R^a_{bac}$. So modulo a sign then there is only one non–zero contraction of the Riemann curvature tensor, which we call the **Ricci tensor**.

$$R_{ab} = R^c_{abc}$$

NB there is no widely accepted convention for the sign of the Riemann curvature tensor, or the Ricci tensor, so check the sign conventions of whatever book you are reading.

The Ricci tensor is a second order tensor about curvature while the stress-energy tensor is a second order tensor about the source of gravity (energy density). So how about $R^{\mu\nu} = T^{\mu\nu}$ as the equation we are after which is curvature=gravity ? This in fact was Einsteins first guess!

well, stress energy tensor is also symmetric, and has covariant derivative of zero. The Ricci tensor is also symmetric. Contract the cyclic identity $R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0$ so

$$R^a_{bca} + R^a_{cab} + R^a_{abc} = 0$$

But we just saw that $R_{aabc} = 0$. We need to raise an a index, but there are 2 a indices so $R^a_{abc} = g^{aa} R_{aabc} = 0$ so

$$R^a_{bca} + R^a_{cab} = 0$$

$$R^a_{bca} - R^a_{cba} = 0 \text{ from } 2$$

$$R_{bc} = R_{cb}$$

so we have two out of the 3 qualities we need for the stress energy tensor. But what about the third - covariant derivative of zero ?

We can use the Bianchi identity to get this for the Ricci tensor

$$R^a_{bca;e} + R^a_{bae;c} + R^a_{bec;a} = 0$$

$$R_{bc;e} - R_{be;c} + R^a_{bec;a} = 0$$

we know (section 3.10) that we can take the metric in and out of covariant differentiation ie that raising and lowering of indices can go in and out of covariant differentiation. So lets raise the index b - multiply it by g^{fb} and then turn the f's into b's again

$$g^{fb}R_{bc;e} - g^{fb}R_{be;c} + g^{fb}R^a_{bec;a} = 0$$

$$R^b_{c;e} - R^b_{e;c} + R^{ab}_{ec;a} = 0$$

contract over e and b ie set $e = b$

$$R^b_{c;b} - R^b_{b;c} + R^{ab}_{bc;a} = 0$$

Define $R = R^a_a$ as the Riemann curvature scalar. Then we have

$$R^b_{c;b} - R_{;c} + R^{ab}_{bc;a} = 0$$

$$R^b_{c;b} - R_{;c} + R^{ba}_{cb;a} = 0 \quad \text{from 3}$$

$$R^b_{c;b} - R_{;c} + R^a_{c;a} = 0$$

$$2R^a_{c;a} - R_{;c} = 0$$

$$(R^a_c - \frac{1}{2}R\delta^a_c)_{;a} = 0$$

$$(g^{bc}R^a_c - \frac{1}{2}g^{bc}\delta^a_c R)_{;a} = 0$$

$$(R^{ba} - \frac{1}{2}g^{ba}R)_{;b} = 0$$

so the covariant derivative of the Ricci tensor IS NOT ZERO! But this funny combination of the ricci tensor and curvature scalar IS.

4.7 The Einstein Field Equations

So we have the Ricci Tensor, which is a symmetric second order tensor, but its divergance IS NOT zero. Instead we have this composite of the Ricci tensor and the curvature scalar which has divergance of zero. So its this whole thing which we could put as proportional to the stress-energy tensor.

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = \kappa T^{\alpha\beta}$$

The mixed form of this equation is

$$g_{\gamma\alpha}R^{\alpha\beta} - \frac{1}{2}g_{\gamma\alpha}g^{\alpha\beta}R = \kappa g_{\gamma\alpha}T^{\alpha\beta}$$

$$R^{\beta}_{\gamma} - \frac{1}{2}\delta^{\beta}_{\gamma}R = \kappa T^{\beta}_{\gamma}$$

contract over β and γ i.e. set $\beta = \gamma$ Then since $\delta^{\beta}_{\beta} = 4$ then

$$R - 2R = \kappa T$$

$$R = -\kappa T$$

So an alternative form is

$$R^{\alpha\beta} = \kappa(T^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}T)$$

These are the Einstein field equations. There are 10 of them - should be 4^2 but the tensors are SYMMETRIC. But this is not a unique way to relate curvature to gravity and there is no way to prove them a priori. In fact this was Einsteins second guess at it - his first was $R^{ab} = \kappa T^{ab}$ And its not the final guess either, even in this form. In effect what we have is a differential equation. So the solution has an integration constant in it. We get the equation above if this constant is zero. But the more general form is

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R + \Lambda g^{\alpha\beta} = \kappa T^{\alpha\beta}$$

where Λ is the cosmological constant.

We can solve for κ by saying that this equation should give us the Newtonian limits when gravity is weak (so $g_{\alpha\beta} \approx \eta_{\alpha\beta} + h_{\alpha\beta}$ and the Christoffel symbols are zero) and the density/pressure of matter is fairly small. The $T^{\mu\nu}$ are then mostly zero except for $T^{00} = \rho_0 c^2$ and $T \approx \rho_0 c^2$. So then we have

$$R^{00} = \kappa(\rho_0 c^2 - \frac{1}{2}\rho_0 c^2)$$
$$R^{00} = \frac{1}{2}\kappa\rho_0 c^2$$

But we know what the Ricci tensor components are in the weak field limit (Christoffel symbols are approx zero and metric approx Minkowski)

$$R^{00} = R_{00} = R^{\alpha}_{00\alpha} \approx \partial_0 \Gamma^{\alpha}_{0\alpha} - \partial_\alpha \Gamma^{\alpha}_{00} \approx -\partial_\alpha \Gamma^{\alpha}_{00}$$

From our previous weak field stuff (lecture 8, section 3.9) we know that $\Gamma^{\alpha}_{00} = -1/2\eta^{ij}\partial j h_{00} = 1/2\delta^{ij}\partial j h_{00}$ so

$$\partial_\alpha \Gamma^{\alpha}_{00} \approx \partial_i \Gamma^{i}_{00} = \partial_i 1/2\delta^{ij}\partial j h_{00} = \frac{1}{2}\delta^{ij}\partial_i\partial_j h_{00} = \frac{1}{2}\partial^i\partial_i h_{00}$$

where $h_{00} = 2\phi/c^2$ and $\phi = -GM/R$ is the gravitational potential. So we have

$$R_{00} = -\frac{1}{2}\delta^{ij}\partial_i\partial_j h_{00} = -\frac{1}{c^2}\nabla^2\phi = \frac{1}{2}\kappa\rho_0 c^2$$

But we also have Poissons equation for Newtonian gravity which is

$$\nabla^2\phi = 4\pi G\rho_0$$

So this works if $\kappa = -8\pi G/c^4$

So the full Einstein field equations are

$$R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = -\frac{8\pi G}{c^4}T^{\alpha\beta}$$

or

$$R^{\alpha\beta} = -\frac{8\pi G}{c^4}(T^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}T)$$

5 The Schwarzschild solution

Now we know how energy density relates to curvature we can use this to get the metric for curved spacetime in various simple geometries. The first one is to look at static curved spacetime round a spherically symmetric mass while the rest of spacetime is empty. Schwarzchild did this by guessing the form it should have

$$c^2d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

so the $g_{\mu\nu}$ are not functions of t - field is static. And spherically symmetric as surfaces with r, t constant have $ds^2 = r^2(d\theta^2 + sin^2\theta d\phi^2)$. While the rest of spacetime being empty says $A(r) \to c^2$ and $B(r) \to 1$ as $r \to \infty$

Then we know the forms for the Christoffel symbols (from homework set 2) are

$$\Gamma^0_{01} = A'/2A$$

$$\Gamma^1_{00} = \frac{A'}{2B} \quad \Gamma^1_{11} = \frac{B'}{2B} \quad \Gamma^1_{22} = -\frac{r}{B} \quad \Gamma^1_{33} = -\frac{r\sin^2\theta}{B}$$

$$\Gamma^2_{12} = \frac{1}{r} \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{13} = \frac{1}{r} \quad \Gamma^3_{23} = \cot\theta$$

For EMPTY spacetime then $R_{\mu\nu} = 0$

$$R_{\mu\nu} = \partial_\nu\Gamma^\sigma_{\mu\sigma} - \partial_\sigma\Gamma^\sigma_{\mu\nu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} = 0$$

Substitute in with $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and get

$$R_{00} = \partial_0\Gamma^\sigma_{0\sigma} - \partial_\sigma\Gamma^\sigma_{00} + \Gamma^\rho_{0\sigma}\Gamma^\sigma_{\rho 0} - \Gamma^\rho_{00}\Gamma^\sigma_{\rho\sigma} = 0$$

$$= -\partial_1\Gamma^1_{00} + \Gamma^\rho_{01}\Gamma^1_{\rho 0} + \Gamma^\rho_{00}\Gamma^0_{\rho 0} - \Gamma^1_{00}\Gamma^\sigma_{1\sigma} = 0$$

$$= -\frac{\partial}{\partial r}(\frac{A'}{2B}) + \Gamma^0_{01}\Gamma^1_{00} + \Gamma^1_{00}\Gamma^0_{10} - \Gamma^1_{00}\Gamma^\sigma_{1\sigma} = 0$$

$$= -\frac{A''}{2B} + \frac{A'B'}{2B^2} + 2\frac{A'}{2A}\frac{A'}{2B} - \Gamma^1_{00}\Gamma^0_{10} - \Gamma^1_{00}\Gamma^1_{11} - \Gamma^1_{00}\Gamma^2_{12} - \Gamma^1_{00}\Gamma^3_{13} = 0$$

$$= -\frac{A''}{2B} + \frac{A'B'}{2B^2} + \frac{A'A'}{2AB} - \frac{A'}{2B}(\frac{A'}{2A} + \frac{B'}{2B} + \frac{1}{r} + \frac{1}{r}) = 0$$

$$= -\frac{A''}{2B} + \frac{A'B'}{4B^2} + \frac{A'A'}{4AB} - \frac{A'}{rB} = 0$$

$$= -\frac{A''}{2B} + \frac{A'}{4B}(\frac{A'}{A} + \frac{B'}{B}) - \frac{A'}{rB} = 0$$

similarly, do the others and get

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B'}{rB} = 0$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}(\frac{A'}{A} - \frac{B'}{B}) = 0$$

$$R_{33} = R_{22}\sin^2\theta = 0$$

All the rest of the $R_{\mu\nu}$ are zero. The R_{33} is redundant information as its linearly related to R_{22} so we have 3 equations for our unknown A and B.

Add $BR_{00}/A + R_{11} = 0$ to get $A'B + AB' = 0$ so $AB =$constant. As $r \to \infty$ then $AB \to c^2$ so $B = c^2/A$. Substitute into R_{22} and get

$$R_{22} = A/c^2 - 1 + A'r/c^2 = 0$$

$$\frac{d(rA)}{dr} = c^2$$

$$rA = c^2(r + k)$$

$$A = c^2(1 + k/r)$$

But in the weak field limit we know that $g_{00} \approx 1 - 2GM/(rc^2)$ (lecture 8) so $k = -2GM/c^2$ and we have

$$A = c^2(1 - 2\frac{GM}{rc^2}) \quad B = (1 - 2\frac{GM}{rc^2})^{-1}$$

so we have the line element

$$ds^2 = c^2 d\tau^2 = c^2(1 - \frac{2GM}{c^2 r})dt^2 - (1 - \frac{2GM}{c^2 r})^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2$$

or set $GM/c^2 = m$

$$ds^2 = c^2 d\tau^2 = c^2(1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2$$

5.1 Meaning of the Schwarzschild metric

We derived the metric from the Einstein equations, assuming that the cosmological constant is negligible and get

$$ds^2 = c^2 d\tau^2 = c^2(1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

The $dt, dr, d\theta, d\phi$ are COORDINATES, not proper time, or proper radius. What do they mean ?

With $m = 0$ then the metric becomes simply Minkowski in spherical coordinates, where r is simply distance from the origin at a fixed time $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$. For fixed θ and ϕ then $ds^2 = dr^2$ When we turn up m then the distances go strange - $ds^2 = (1 - 2m/r)^{-1}dr^2$. The proper distance ds IS NO LONGER given by dr, instead we have $ds > dr$. The way to do this is to curve space.

Now for time, with $m = 0$ then a fixed point in space has $c^2 d\tau = c^2 dt^2 = ds^2$, so $d\tau = dt$ and t is the time measured by clocks which are stationary in the reference frame. Clocks record proper time intervals along their world lines - this clock is stationary its proper time is the same as coordinate time. Now turn m up, and $d\tau^2 = (1 - 2m/r)dt^2$. We are still going to say that clocks record proper time intervals along their world lines, and this clock is fixed, so we have that coordinate time is NOT equal to proper time $d\tau < dt$.

Asymptotically, both r and t go to proper distance and proper time. But these COORDINATES don't measure proper distance and proper time in a strong gravitational field. Coordinates are usefula s they show us how to connect things happening at one bit of spacetime to things happening at another point. But they themselve are NOT physical measures of proper time or distance.

What we are going to do is find the geodesics (inertial frame) paths in this metric.

5.2 Gravitational redshift: radial null geodesics for photons

We send a light beam along a radial null geodesic. so the metric becomes

$$ds^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1}dr^2 = 0$$

becuase light travels on NULL geodesics. The beam travels at coordinate speed

$$\frac{dr}{dt} = \pm c(1 - 2m/r)$$

so the coordinate time taken to go from the radius where its emitted r_E to where its recieved r_R (use +ve if going outwards)

$$t_R - t_E = \frac{1}{c}\int_{r_E}^{r_R} (1 - 2m/r)^{-1} dr$$

we could evaluate this but we're not going to as the important this is that it only depends on space. If we send another signal a bit later from the same fixed position emitter to the same fixed position reciever then this coordinate time difference will be the same.

$$t_{R1} - t_{E1} = t_{R2} - t_{E2}$$

so then the coordinate time difference between the two emitted signals and the two received signals is also the same

$$t_{R1} - t_{R2} = t_{E1} - t_{E2} = \Delta t$$

But the clock at the point of emission records proper time not coordinate time. It is fixed in space, so the time it measures between the two signals is

$$\Delta\tau_E = (1 - 2m/r_E)^{1/2}\Delta t_E = (1 - 2m/r_E)^{1/2}\Delta t$$

but so is the clock at the receiver

$$\Delta\tau_R = (1 - 2m/r_R)^{1/2}\Delta t_R = (1 - 2m/r_R)^{1/2}\Delta t$$

so

$$\frac{\Delta\tau_E}{\Delta\tau_R} = \Big(\frac{1 - 2m/r_E}{1 - 2m/r_R}\Big)^{1/2}$$

Suppose the emitter were pulsating, then it has frequency $\nu \propto 1/\Delta\tau$. So

$$\frac{\nu_R}{\nu_E} = \Big(\frac{1 - 2m/r_E}{1 - 2m/r_R}\Big)^{1/2}$$

If $r_E < r_R$ then the photon crawls uphill so it loses energy so $\nu_R < \nu_E$. This can be measured - in the laboratory with a large tower (Pounds–Rebka–Snyder experiment - tiny effect). Or look at absorption lines from the surface of a white dwarf and get a much bigger effect. it works!

5.3 Particle Geodesics - elliptical orbits

We've just seen an observational test of GR. What about some more ? Well, one thing we could do is look at orbits and see if there is any observavational tests between GR and Newtonian gravity. Planet orbits can be measured very precisely, so there is a potential test here. These orbits are for massive particles (so not NULL) and they are ELLIPTICAL - vary in both ϕ and r with respect to time. Orbits are geodesic paths so must satisfy the Euler-Lagrange equation where $L = 1/2 g_{\alpha\beta}\dot{x^\alpha}\dot{x}^\beta$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$$

where dot denotes derivative with respect to proper time (which is an affine parameter).

First thing to do is write down the metric - tailor it to the situation you want to solve. Here we want particle elliptical orbits. These change in both ϕ and r but they are in a PLANE (θ=constant) so without loss of generality as the metric is spherically symmetric we can take this plane to be $\theta = \pi/2$ so the metric is then

$$c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2 d\phi^2$$

The easy equations will be those where a coordinate does not appear e.g. t and ϕ as then the Euler Lagrange equations only have one term

$$\frac{d}{d\tau}\Big(\frac{\partial L}{\partial \dot{x}^\alpha}\Big) = 0$$

so $\partial L/\partial \dot{x}^\alpha$ = constant. These are called cyclic coordionates. So for t

$$\frac{\partial L}{\partial \dot{t}} = c^2(1 - 2m/r)\dot{t} = \text{constant} = k$$

Similarly for ϕ.

$$\frac{\partial L}{\partial \dot{\phi}} = -r^2\dot{\phi} = \text{constant} = h$$

So we have already relations for $\dot{\phi}$ and $\dot{t}$. All we need now is one for $\dot{r}$ and we'd have everything we need to do elliptical orbits. We can get a relation for $\dot{r}$ from the Euler-Lagrance equations but its a bit nasty. All we really need is some way to relate $\dot{r}$ to the known $\dot{\phi}, \dot{t}$, and we can get such a relation more easily through the METRIC

$$c^2 = c^2(1-2m/r)\dot{t}^2 - (1-2m/r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$

rearrange to get

$$c^2(1-2m/r)\frac{\dot{t}^2}{\dot{\phi}^2} - (1-2m/r)^{-1}\frac{dr}{d\phi} - r^2 = \frac{c^2}{\dot{\phi}^2}$$

substitute in from the EL equations for $\dot{t}, \dot{\phi}$,

$$\frac{r^4k^2c^2}{(1-2m/r)h^2} - (1-2m/r)^{-1}\left(\frac{dr}{d\phi}\right)^2 - r^2 = \frac{r^4c^2}{h^2}$$

multiply by $-(1-2m/r)$

$$\left(\frac{dr}{d\phi}\right)^2 + r^2(1+c^2r^2/h^2)(1-2m/r) - \frac{k^2c^2r^4}{h^2} = 0$$

let $u = 1/r$ so $dr/d\phi = dr/du du/d\phi = -1/u^2 du/d\phi$

$$\frac{1}{u^4}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{u^2}(1+\frac{c^2}{h^2u^2})(1-2mu) - \frac{k^2c^2}{h^2u^4} = 0$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2(1+\frac{c^2}{h^2u^2})(1-2mu) - \frac{k^2c^2}{h^2} = 0$$

$$\left(\frac{du}{d\phi}\right)^2 + (u^2+\frac{c^2}{h^2})(1-2mu) - \frac{k^2c^2}{h^2} = 0$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - 2mu^3 + \frac{c^2}{h^2} - \frac{2mc^2u}{h^2} - \frac{k^2c^2}{h^2} = 0$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2}{h^2}(k^2-1) + \frac{2mc^2u}{h^2} + 2mu^3$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = E + \frac{2GMu}{h^2} + \frac{2GMu^3}{c^2}$$

c.f. the Newtonian orbit analogue

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = E + \frac{2GMu}{h^2}$$

SO GEODESICS IN CURVED SPACETIME GIVE ORBITS LIKE NEWTONIAN ORBITS, BUT WITH SLIGHT DIFFERENCES - TESTABLE!!!!!!

5.4 Advance of Perihelion of Mercury: weak field

E is a constant related to the energy of the orbit. $du/d\phi = 0$ at perihelion and aphelion. so newtonian goes to

$$u^2 = E + \frac{2GMu}{h^2}$$

solve this to

$$u = \frac{GM}{h^2} \pm \sqrt{\frac{G^2M^2}{h^4} + E}$$

let $y = u - GM/h^2$ i.e. we are looking for small deviations about the mean distance. Substitute in for $u = y + GM/h^2$ and $du/d\phi = dy/d\phi$ so

$$(\frac{dy}{d\phi})^2 + y^2 + \frac{2GMy}{h^2} + \frac{G^2M^2}{h^4} = E + \frac{2GMy}{h^2} + \frac{2G^2M^2}{h^4}$$

$$(\frac{dy}{d\phi})^2 + y^2 = E + \frac{G^2M^2}{h^4}$$

Trial solution of $y = A\cos(\phi - \phi_0)$, so $dy/d\phi = -Asin(\phi - \phi_0)$ and $(dy/d\phi)^2 = A^2sin^2(\phi - \phi_0) = A^2 - y^2$ so it works and we could solve for A if we wanted. But the key thing here is that these small deviations from circularity which make the orbit an ellipse are periodic on 2π. The orbit comes back to the same r after 2π so its CLOSED. Now lets do the SAME THING, but with that extra GR term in u^3.

$$(\frac{dy}{d\phi})^2 + y^2 = E + \frac{G^2M^2}{h^4} + \frac{2GM}{c^2}(y + GM/h^2)$$

$$(\frac{dy}{d\phi})^2 + y^2 = E + \frac{G^2M^2}{h^4} + \frac{2GM}{c^2}(y^3 + \frac{3GM}{h^2}y^2 + \frac{3G^2M^2}{h^4}y + \frac{G^3M^3}{h^6})$$

neglect terms in y^3 - these are SMALL deviations.

$$(\frac{dy}{d\phi})^2 + (1 - \frac{6G^2M^2}{h^2c^2})y^2 = E + \frac{G^2M^2}{h^4} + \frac{2G^4M^4}{h^6c^2} + \frac{6G^3M^3y}{h^4c^2}$$

The trial solution here is $y = y_0 + A\cos(k\phi - \phi_0)$ which has $dy/d\phi = -Ak\sin(k\phi - \phi_0)$ and $(dy/d\phi)^2 = A^2k^2\sin^2(k\phi - \phi_0) = A^2k^2 - k^2y^2$, This is going to work as we now have a multiplier for y^2. Equating coefficients for y^2 gives

$$-k^2 + (1 - \frac{6G^2M^2}{h^2c^2}) = 0$$

$$k = (1 - \frac{6G^2M^2}{h^2c^2})^{1/2}$$

So we get round to the same r NOT when $\phi = 2\pi$ but when $k\phi = 2\pi$ ie at

$$\phi = \frac{2\pi}{k} = 2\pi(1 - \frac{6G^2M^2}{h^2c^2})^{-1/2} \approx 2\pi(1 + \frac{3G^2M^2}{h^2c^2})$$

from a binomial expansion. Then the change in ϕ in one orbit is

$$\Delta\phi = \phi - 2\pi = 6\pi G^2M^2/(h^2c^2)$$

So orbits in GR are not (quite) closed. An elliptical orbit will have the closest point precess around. So first we need this consstant $h = r^2\dot{\phi} = r^2 2\pi/P$ where P is the period. In Newtonian this is $P^2 = 2\pi r^3/(GM)$ so $h^2 = r^4GM/r^3 = GMr$ so $\Delta\phi = 6\pi G^2M^2/(GMrc^2) = 6\pi GM/(rc^2)$. For Mercury (closest planet so one in the strongest gravitational field, ie where this effect will be biggest) then $r \sim 5.2 \times 10^{10}m$ so $\Delta\phi = 5 \times 10^{-7}$ radians per orbit. This is TINY!! but its also cumulative. One orbits precession is difficult to measure, but after 100 years ie 415 orbits then the point of closest approach (perihelion) has shifted by 0.01189 radians or 43 arcseconds. This is measurable, and was a well known problem in the 19th century that newtonian mechanics didn't get the right answer for Mercury's orbit. GR does!!!

5.5 More on Null geodesics

We looked at radial null geodesics to get gravitational time dilation. Now lets look at the angular dependence of the null geodesic equations. We are especially interested in ϕ paths as this will give us lightbending.

The Lagrangian is still the same but we have to say that dots now denote derivative with respect to some affine parameter which is not proper time (as thats zero!). So we can use our previous solutions of the EL equations for $\dot{t}$ and $\dot{\phi}$, but we have to change the third equation as we took it straight from the metric, which is now zero! Again lets set $\theta = \pi/2$ and $d\theta = 0$ ie we are working in a plane.

$$(1 - 2m/r)\dot{t} = k \quad r^2\dot{\phi}^2 = h$$
$$c^2(1 - 2m/r)\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0$$

substitute the top two into the third and divide by $\dot{\phi}^2$ and we get

$$\frac{c^2(1 - 2m/r)k^2r^4}{(1 - 2m/r)^2h^2} - \frac{1}{(1 - 2m/r)}\Big(\frac{dr}{d\phi}\Big)^2 - r^2 = 0$$

$$\frac{c^2k^2r^4}{h^2} - \Big(\frac{dr}{d\phi}\Big)^2 - (1 - 2m/r)r^2 = 0$$

Again let $u = 1/r$

$$\frac{1}{u^4}\Big(\frac{du}{d\phi}\Big)^2 + \frac{1}{u^2}(1 - 2mu) = \frac{c^2k^2}{u^4h^2}$$

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = \frac{c^2k^2}{h^2} + \frac{2GM}{c^2}u^3$$

5.6 Light bending: weak field limits

Again we have a u^3 term which can be thought of as a GR correction. Without it we have ($F = c^2k^2/h^2$)

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = F$$

which has a particular solution of $u = u_0 \sin\phi$ where u_0 is the value of u at closest approach. Put this in terms of r and get $r\sin\phi = 1/u_0 = r_0$. This is just a straight line. This is what Newtonian gravity predicts about light! Now lets do the weak field approximation again

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = F + \epsilon u^3$$

at closest approach then $u = u_0$ again and $du/d\phi = 0$ so

$$u_0^2 = F + \epsilon u_0^3$$

we can use this to substitute for F in the general equation

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = u_0^2(1 - \epsilon u_0) + \epsilon u^3$$

This should have a solution which is just a small perturbation of the flat spacetime solution - maybe $u = u_0 \sin\phi + \epsilon f(\phi)$ where f is some function of ϕ to be determined. Stick this in, and ignore anything with higher powers of ϵ

$$2\cos\phi \frac{df}{d\phi} + 2f \sin\phi = u_0^2(\sin^3\phi - 1)$$

This is just a first order differential equation of f, so we'll go look up the solution.

$$f = \frac{1}{2}u_0^2(1 + \cos^2\phi - \sin\phi) + A\cos\phi$$

A is a constant of integration so can be found from the boundary conditions. At $r = \infty$ then $\phi = 0$ and $f = 0$, so $A = -u_0^2$. So our full solution for the weak field effect of gravity on the path of a light beam is

$$u = u_0(1 - \frac{1}{2}\epsilon u_0)\sin\phi + \frac{1}{2}\epsilon u_0^2(1 - \cos\phi)^2$$

So our full solution for the weak field effect of gravity on the path of a light beam is

$$u = u_0(1 - \frac{1}{2}\epsilon u_0)\sin\phi + \frac{1}{2}\epsilon u_0^2(1 - \cos\phi)^2$$

so we no longer expect a straight line as the photon goes from $\phi = 0$ to π. As $r \to \infty$ i.e. $u \to 0$ then $\phi \to \pi + \alpha$ where $\alpha \ll 1$. $\cos(\pi + \alpha) = \cos\pi\cos\alpha - \sin\pi\sin\alpha \approx -1$ while $\sin(\pi + \alpha) = \sin\pi\cos\alpha + \cos\pi\sin\alpha = -\sin\alpha \approx -\alpha$.

$$0 = -(1 - \frac{1}{2}\epsilon u_0)\alpha + 2\epsilon u_0 \approx -\alpha + 2\epsilon u_0$$

$$\alpha \sim 2\epsilon u_0 = 4\frac{GM}{r_0 c^2}$$

Take $r_0 = 1/u_0$ as equal to the suns radius and look at a star in the line of sight just away from the limb of the sun and then the prediction for the deflection of its position is 1.75 seconds of arc. This was the FIRST test of GR, measuring positions of stars close to the sun as seen during a solar eclipse and comparing these to their positions as seen when the sun was nowhere near the line of sight. newtonian gravity only affects MASS. GR affects the path of anything that travels over curved spacetime.

5.7 Circular orbits in Strong Gravity

We've seen that GR works in the wimpy weak field approximations. so lets see what it predicts in strong fields..... We had the full orbit equations in section 5.3, but this was allowing elliptical orbits. We'll thats a bit nasty really. lets go instead to circular ones. Then our metric simplifies a bit as $dr = d\theta = 0$

$$ds^2 = c^2 d\tau^2 = c^2(1 - 2m/r)dt^2 - r^2 d\phi^2$$

so the Lagrangian is $L = 1/2(c^2(1-2m/r)\dot{t}^2 - r^2\dot{\phi}^2)$ and the Euler Lagrange equations for t and r (cyclic coordinates) are as before

$$(1 - 2m/r)\dot{t} = k \qquad r^2\dot{\phi} = h$$

But now we can also do the Euler lagrange equations for r to get

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

$$\frac{\partial L}{\partial r} = \frac{1}{2}c^2\dot{t}^2\frac{2m}{r^2} - r\dot{\phi}^2 0 = \frac{mc^2\dot{t}^2}{r^2} - r\dot{\phi}^2 = 0$$

$$mc^2\dot{t}^2 = r^3\dot{\phi}^2$$

so COORDINATE time for one orbit is

$$mc^2 t_{orb}^2 = r^3(2\pi)^2 \qquad t_{orb} = 2\pi(r^3/GM)^{1/2}$$

ie same as Newtonian.

PROPER TIME measured by someone going round on this orbit (sub. back into metric)

$$c^2 = c^2(1 - 2m/r)\dot{t}^2 - r^2\dot{\phi}^2 = c^2(1 - 2m/r)\dot{t}^2 - r^2 mc^2\dot{t}^2/r^3$$

$$1 = (1 - 3m/r)(dt/d\tau)^2$$

$$d\tau = (1 - 3m/r)^{1/2}dt$$

$$\tau_{orb} = (1 - 3m/r)^{1/2}t_{orb}$$

NB there are NO geodesic circular orbits possible at $r < 3m$. This is very different to Newtonian gravity where circular orbits are always possible. In newtonian orbits you can always balance gravity just by running round faster.

In SR/GR you can only run as fast as the speed of light. This is the orbit for going at the speed of light. So there are no faster orbits. And if you were going around on this orbit, then you wouldn't age.

PROPER TIME as measured by someone hovering at some radius r_0 - using rockets (not on a geodesic) if near the BH. Suppose the orbiting ship sends a signal each time it completes 1 orbit, and that this signal travels on a radial null geodesic. From our gravitational redshift we know that coordinate time intervals are the same measured at all radii and that these relate to proper time intervals as $d\tau = (1 - 2m/r_0)^{1/2} dt$. So this gives us a way to relate a coordinate time difference measured at one radius with proper time at another. So orbital time measured at radius r_0 is $\tau_{orb}(r_0) = t_{orb}(r)(1 - 2m/r_0)^{1/2} \to t_{orb}$ as $r_0 \to \infty$

5.8 Nature of the Event Horizon

Go into a black hole on a radial free fall geodesic. How long does it take to fall from r_1 to r_2 where $r_1 \gg 2m$ Radial geodesic with $d\theta = d\phi = 0$

$$c^2 = c^2(1 - 2m/r)\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2$$

Substitute in from the particle EL equation for t, $(1 - 2m/r)\dot{t} = k$ and we get

$$\dot{r}^2 - c^2k^2 + c^2(1 - 2m/r) = 0$$

i.e. k depends on which geodesics we are on. For a particle at rest at $r = r_1 >> 2m$ then

$$k^2 = (1 - 2m/r_1) \approx 1$$

Substitute this back into our free fall geodesic and we have

$$\dot{r}^2 - c^2 + c^2(1 - 2m/r) = 0$$

$$\dot{r} = \pm c(2m/r)^{1/2}$$

We have to take a square root so this only makes sense if $r < r_1$ as is expected as a particle needs to fall DOWN if its on a geodesic. Then we can have either sign for $\dot{r}$, and again since particles fall down then it makes most sense to take the -ve one so

$$\dot{r} = \frac{dr}{d\tau} = -\sqrt{2GM}\left(\frac{1}{r}\right)^{1/2}$$

It looks the same as the standard Newtonian expression but here we are taking the derivative with respect to proper time τ NOT coordinate time t, and the radial coordinate is NOT the same as radial distance.

So we can now calculate how long it would take to fall in the gravitational field. $\tau = \int_{r_1}^{r_2} d\tau/dr dr$ so

$$\tau = \frac{1}{c\sqrt{2m}} \int_{r_2}^{r_1} r^{1/2} dr = \frac{2}{3c\sqrt{2m}} (r_1^{3/2} - r_2^{3/2})$$

so the proper time (time as measured by a clock on this worldline) taken to fall from R to r_2 is FINITE - even if $r_2 < 2m$. its smoothly varying across $r = 2m$ and we can take r_2 down to zero! But coordinate time is nothing like so nice. The Euler lagrange equation was $(1 - 2m/r)\dot{t} = k \sim 1$ for a particle starting from rest so $dt/d\tau = 1/(1 - 2m/r)$ so the coordinate speed is

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = -c(\frac{2m}{r})^{1/2}(1 - 2m/r)$$

This drops to zero at $r = r_2$! so it takes an infinite amount of coordinate time to fall to $r = 2m$!! We saw from the redshift argument that coordinate time intervals between 2 photons emitted at 1 radius are the same as the coordinate time intervals between 2 photons received at another radius, and also that at $r \to \infty$ then $dt \to d\tau$ so to an observer sitting a long way away the person falling into the black hole gets slower and slower as they approach the event horizon. They appear to hover on the horizon for ever, while any light/signals emitted from them gets infinitely redshited as $r_2 \to r_s$. The redshift (ie observed frequency) goes imaginary for $r_2 < r_s$. But $E = h\nu$ so no light/energy can get out from within $r = 2m$.

Now to some fun! what happens at $r = 3m, 2m, 0$? First look at the coordinate speed of light. From the metric this is

$$\frac{dr}{dt} = \pm c(1 - 2m/r)$$

as $r \to 2m$ then $dr/dt \to 0$. In these coordinates the light cones 'close up' as $r \to 2m$, and it takes an infinite amount of time (as measured by t which is the proper time for a distant observer!!) for the light to get out from close to $r = 2m$. Similarly, from INSIDE $r = 2m$ then it takes an infinte amount of (coordinate) time for light to get to $r = 2m$. So light signals cannot propagate to infinity from within $r = 2m$ so its termed an event horizon - its a peculiar kind of coordinate singularity that constitutes a one-way boundary between the interior and exterior of a black hole. The region inside is in some sense disconnected from the rest of the universe its tiny (1cm for the Earth, 3000 m for the Sun)!

But what's REALLY happening?? Well space is DRAMATICALLY curved. And if time is orthogonal to space then the time axis gets tilted too. The light cones still make angles of 45 degrees to a local time-space axis (remember we can always make LOCALLY flat spacetime, even on the horizon and beyond!!) But the light cone can be tilted over beyond 45 degrees, at which point no light can escape to infinity. You can see this explicitally in Eddington-Finkelstein coordinates, but we won't do them here. So the REAL singularity at $r = 0$ (which may or may not be a true singularity - we need quantum gravity to follow what happens at such strong gravity on such tiny scales) is shielded from our view by this odd coordinate singularity.

6 Cosmology

6.1 Robertson-Walker metric

We assume the cosmological principle, which is that that at a given time t the universe is homogeneous (constant density everywhere), and isotropic (the same in all directions). So the view from earth is a typical view of the universe.

The hot Big Bang model of the universe is that the universe started with an event in which space/time/matter/energy were explosively created about 16 billion years ago. This explains

1) Hubbles law - distant galaxies are receeding from us with velocity approximately proportional to distance. Spacetime is still expanding and carrying matter (galaxies) along with it. Redshift is defined as $z = (\lambda_{\rm obs} - \lambda_{\rm emm})/\lambda_{\rm emm}$. Hubble law is $z = H_0 d/c$ for nearby galaxies, with $H_0 \sim 50$ km s^1 Mpc^{-1}

2) microwave background radiation. Now at 2.7 K, this is a relic of the hot early universe. Hot dense material emitts as a blackbody. when Hydrogen recombines at 3000 K then the photons no longer easily interact with the electrons, so they decouple. and get expanded along with the universe. To

get a 3000 K blackbody to a 3K one needs $z \sim 1000$

3) elemental abundances. if the universe were hot enough then you can get nucleosynthesis fusing H to He. but the universe is expanding and cooling so there is only a short time where the universe is hot enough for fusion but cool enough for H to exist. So only do LIGHT element nucleosynthesis. Explains why we see 25 per cent He.

So we need a metric which can describe NON-STATIC spacetime, where the stress energy tensor components are NON ZERO (universe is filled with perfect fluid). Something like universal time exists in an isotropic universe. Take a set of observers (called fundamental observers) in different locations who are at rest w.r.t the local matter e.g. sitting on different galaxies ad receeding from each other. Global or cosmic time t is time as measured by the clocks of these observers i.e. t is the proper time measured by an observer at rest with respect to the local matter distribution. The coordinate is global as these clocks can be synchronised, e.g. by exchanging light signals to say that the clocks should be set to zero when the local density reaches some agreed value.

So we want a metric which has spherical symmetry in which the spatial part can expand. The easiest way to think about this is with a balloon - as it is blown up then the 2D surface expands. We see this embedded in our 3D space. We want 3D space to expand so the simplest way to do this is to embed it in 4D space. If we were looking at a 4D hypersphere where the curvature was constant and depended only on time then

$$x^2 + y^2 + z^2 + w^2 = R^2(t)$$

$$w^2 + r^2 = R^2(t)$$

differentiate and $2wdw + 2rdr = 0$ so $wdw = -rdr$ so

$$w^2 dw^2 = r^2 dr^2$$

$$dw^2 = \frac{r^2}{R^2(t) - r^2} dr^2$$

so the separation of points on the 4D hypersphere is given by

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + dw^2$$

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{r^2}{R^2(t) - r^2} dr^2$$

$$dl^2 = \frac{R^2(t)}{R^2(t) - r^2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$dl^2 = R^2(t)\Big[\frac{d(r/R)^2}{1-(r/R)^2} + (r/R)^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]$$

Let $\tilde{r} = r/R$ then

$$dl^2 = R^2(t)\Big[\frac{d\tilde{r}^2}{1-\tilde{r}^2} + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]$$

Now convert this to a surface of constant negative curvature by considering an imaginary radial scale factor $R \to iR$ (so $\tilde{r} \to -i\tilde{r}$, and $d\tilde{r} \to -id\tilde{r}$) so

$$dl^2 = R^2(t)\Big[\frac{d\tilde{r}^2}{1+\tilde{r}^2} + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]$$

We can combine these two forms by

$$dl^2 = R^2(t)\Big[\frac{d\tilde{r}^2}{1-k\tilde{r}^2} + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]$$

where k represents the curvature -1 is negatively curved space, 0 is flat (euclidean) space and 1 is positively curved space. Extrapolate this form to get the total Roberston–Walker (hereafter RW) line element by using the equivalence principle that it has to look like SR locally. So

$$ds^2 = c^2dt^2 - R^2(t)\Big[\frac{d\tilde{r}^2}{1-k\tilde{r}^2} + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]$$

We will always now call $r = \tilde{r}$, but this means that r is the COMOVING radial coordinate rather than absolute radial coordinate. Two galaxies which are stationary with respect to the expansion have fixed coming coordinates so $c^2d\tau^2 = ds^2 = c^2dt^2$ ie again we have that coordinate/cosmic time is proper time. Such objects in fixed comoving coordinates follow geodesics as $\Gamma^i_{00} = 0$ in this metric (there are no cross spatial/time terms $dx^i dt$). Geodesics are

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{\nu\mu}\frac{dx^\mu}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

But these objects are in fixed comoving coordinates so $dx^1 = dx^2 = dx^3 = 0$ so the sum reduces to

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = \frac{d^2x^i}{d\tau^2} = 0$$

NB This is the same metric as you have been using in Astrophysics II, but there $R(t)$ is called $a(t)$.

6.2 Einstein Equations

So now what we need to do is use the Einstein equations to figure out what $R(t)$ is, in the same way that we did for the Schwartzchild metric to figure out $A(r)$ and $B(r)$ were when we wrote down the simplest spherically symmetric, static metric as $ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$. So we use the metric to form a Lagrangian and then use the Euler Lagrange equations to find all the non zero Christoffel symbols. I gave you all the non zero Christoffel symbols in the homework question, and got you to use these to calculate the components of the Ricci curvature tensor (contracted Riemman curvature tensor).

The Einstein equations are

$$R_{\mu\nu} = \frac{8\pi G}{c^2}(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})$$

ASSUMING that the cosmological constant $\Lambda = 0$. We ignored this for our Schwarzchild metric, and the GR tests in the solar system and elsewhere show that this must be very small. But it may not be identically zero, and if it is not then it can easily be big enough to dominate over the huge distances of the Universe. This is an area of intense and active current research. But we're going to assume its zero. Then the homework got you to calculate the stress energy tensor components for a universe made up of a perfect fluid. And you set these equal to the Ricci tensor and get the **Friedmann equation**

$$\dot{R}^2 + kc^2 = (8\pi G/3)\rho R^2$$

The derivative of this with respect to t then gives the **continuity equation** (after substituting in for $\ddot{R}$ from the Einstein equations - again this was in the homework)

$$\dot{\rho} + (\rho + p/c^2)(3\dot{R}/R) = 0$$

We can simplify this still further if we assume that the Universe is matter dominated rather than pressure dominated - ie that the pressure is negligible which we termed DUST. Then the continuity equation becomes

$$\dot{\rho} + 3\rho\frac{\dot{R}}{R} = 0$$

The solution to this is $\rho R^3 =$ constant, or $\rho R^3 = \rho_0 R_0^3$ where the subscript 0 denotes present day $t = t_0$ values. So the Friedman equation becomes

$$\dot{R}^2 + kc^2 = (8\pi G/3)\rho R^2 = (8\pi G/3)\rho_0 \frac{R_0^3}{R} = \frac{A^2}{R}$$

and at the present day this implies

$$H_0^2 R_0^2 + kc^2 = (8\pi G/3)\rho_0 R_0^2$$

$$\frac{k}{R_0^2} = \frac{8\pi G}{3c^2}\rho_0 - \frac{H_0^2}{c^2} = \frac{8\pi G}{3c^2}(\rho_0 - \rho_c)$$

where $H(t) = \dot{R}(t)/R(t)$ so $H_0 = \dot{R}_0/R_0$ and $\rho_c = 3H_0^2/(8\pi G)$ is the critical density. We can define the density parameter $\Omega = \rho/\rho_c$. If $\Omega > 1$ then there is more matter than critical and $k > 0$ and the universe has positive curvature. If $\Omega = 1$ then $k = 0$ and the universe is flat. If $\Omega < 1$ then $k < 0$ and the universe has negative curvature.

6.3 Size of the Universe

Lets look at the scale factor changing with time. $\dot{R}^2 + kc^2 = A^2/R$:
k=0 then $\rho_0 = \rho_c$ and $\dot{R} = AR^{-1/2}$ so $dRR^{1/2} = Adt$ and this integrates to

$$\frac{R^{3/2}}{3/2} = At$$

$$R(t) = (3A/2)^{2/3} t^{2/3}$$

Note that $\dot{R} \to 0$ as $t \to \infty$ so its infinite - the scale factor keeps on expanding. The Universe is infinite
k=1 then $\rho_0 > \rho_c$ and

$$\dot{R}^2 + c^2 = \frac{A^2}{R}$$

$$\dot{R} = \left(\frac{A^2 - c^2 R}{R}\right)^{1/2}$$

$$\left(\frac{R}{A^2 - Rc^2}\right)^{1/2} dR = dt$$

set $R = A^2/c^2 \sin^2(\psi/2)$ then $dR = A^2/c^2 \sin(\psi/2)\cos(\psi/2)d\psi$

$$\int_0^\psi \Big(\frac{A^2/c^2 \sin^2(\psi/2)}{A^2 - c^2 A^2/c^2 \sin^2(\psi/2)}\Big)^{1/2} A^2/c^2 \sin(\psi/2)\cos(\psi/2)d\psi = \int_0^t dt$$

$$\int_0^\psi \frac{A/c \sin(\psi/2)}{A(1-\sin^2(\psi/2))^{1/2}} A^2/c^2 \sin(\psi/2)\cos(\psi/2)d\psi = t$$

$$\frac{A^2}{c^3}\int_0^\psi \sin^2(\psi/2)d\psi = t$$

$$\frac{A^2}{2c^3}\int_0^\psi (1-\cos\psi)d\psi = t$$

$$\frac{A^2}{2c^3}(\psi - \sin\psi) = t$$

so we have parametric equations for $R = \frac{A^2}{2c^2}(1-\cos\psi)$ and $t = \frac{A^2}{2c^3}(\psi - \sin\psi)$ This is the equation of a cycloid. Its CLOSED. The universe expands to some maximum value of R and then recollapses.
k=-1 then $\rho_0 < \rho_{0c}$ and

$$\dot{R}^2 = \frac{A^2 + c^2 R}{R}$$

$$\Big(\frac{R}{A^2 + c^2 R}\Big)^{1/2} dR = dt$$

set $R = A^2/c^2 \sinh^2(\psi/2)$ then we do the same thing and end up with $R = \frac{A^2}{2c^2}(\cosh\psi - 1)$, $t = \frac{A^2}{2c^3}(\sinh\psi - \psi)$. As $t \to \infty$ then $\dot{R} \to c^2$ so again this universe is infinite.

So we see that 'all' we need is to measure $\dot{R}_0/R_0 = H_0$ and the current mean density of the universe around us, and then we have everything we need to completely determine the expansion of the universe IF we live in a universe where the curvature is dominated by matter (as opposed to a cosmological constant/vacuum energy term). $\rho_0 - \rho_c$, where $\rho_c = 3H_0^2/(8\pi G)$ gives us the sign of the curvature, k, and then we use

$$\frac{k}{R_0^2} = \frac{8\pi G}{3c^2}(\rho_0 - \rho_c)$$

to get R_0, and then we have everything we need to calculate $R(t)$.

6.4 Observations: Hubble law and cosmological redshift

To do this we need to measure H_0. We do this by the redshift. Take two galaxies. Their COMOVING separation is the same irrespective of time if all that is happening is that they are carried along with the expansion. Sit on one of them, define this as the origin with $(r, \theta, \phi, t) = (0, \theta, \phi, t_0)$. The other galaxy is at (r_e, θ, ϕ, t_e).

The light between them travels on a radial null geodesic.

$$0 = c^2 dt^2 - \frac{R^2(t) dr^2}{(1 - kr^2)^{1/2}}$$

$$\int_{t_e}^{t_0} c \frac{dt}{R(t)} = \int_{r_e}^{0} \frac{dr}{(1 - kr^2)^{1/2}}$$

suppose that was the first wavecrest. The second wavecrest leaves at time $t_e + \delta t_e$ and arrives at $t_0 + \delta t_0$. But the galaxy is still at the same comoving coordinates so

$$\int_{t_e + \delta t_e}^{t_0 + \delta t_0} c \frac{dt}{R(t)} = \int_{r_e}^{0} \frac{dr}{(1 - kr^2)^{1/2}}$$

The spatial part is the same as the light travel paths are the same if the δt's are both so small at $R(t)$ varies very little in that time. So

$$\int_{t_e + \delta t_e}^{t_0 + \delta t_0} c \frac{dt}{R(t)} - \int_{t_e}^{t_0} c \frac{dt}{R(t)} = 0$$

$$= \int_{t_0}^{t_0 + \delta t_0} c \frac{dt}{R(t)} - \int_{t_e}^{t_e + \delta t_e} c \frac{dt}{R(t)}$$

so

$$\frac{\delta t_0}{R_0} = \frac{\delta t_e}{R(t_e)}$$

The time between sucessive wave crests gives us the wavelength

$$\frac{\delta t_0}{\delta t_e} = \frac{\lambda_0}{\lambda_e} = \frac{R_0}{R(t_e)} = 1 + z(t_e) = \frac{\nu_e}{\nu_0}$$

At small distances we have Hubbles law. Proper distance is physical distance over the curved spacetime with both ends measured at the same time, so

$$d_{pr} = R(t_0) \int_{r_e}^{0} \frac{dr}{(1 - kr^2)^{1/2}}$$

For small distances then this is $d_{pr} \approx R_0 r_e$, so

$$v_{pr} \approx \dot{R}_0 r_e = \dot{R}_0 \frac{d_{pr}}{R_0} = H_0 d_{pr}$$

So all we need to do is use some sort of standard candle (very bright stars such as cepheids) to get distance, and then we can MEASURE $H_0 \sim 60 - 70$ km s^{-1} Mpc^{-1} (yes, I know they are vicious astronomer units). And this give ρ_c.

Combine this with a MEASURE of ρ_0 and there were have EVERYTHING. we get that $\rho_0/\rho_c = \Omega \sim 0.3$ so the curvature is negative. And we then know $R(t)$, $\dot{R}$ and $\ddot{R}$ etc.

6.5 Deceleration parameter

Yes, but now how do we test this ? The universe is currently expanding, but gravity is slowing it down. So if we looked back into the past the galaxies would be expanding away from each other faster. Because light travels at a finite speed then looking to large distances is the same as looking back in time. So we can look for this to test our models. We used $\dot{R}$ to get H_0, but $\ddot{R}$, a measure of the deceleration, was NOT part of the things we used to get $R(t)$. So we can use this to TEST our assumptions (matter dominated universe, Einstein gravity). And when we do this (its difficult) we get that current results clearly indicate that the expansion of the Universe is SPEEDING UP (accelerating) NOT decelerating.

6.6 Conclusions

So there we are. A universe with curvature dominated by matter (including dark matter) and with gravity given by Einsteins equations should be decelerating. But it appears instead to be ACCELERATING. This (along with other some other bits of evidence) makes us believe that the universe is actually NOT matter dominated, but instead is dominated by vacuum energy from a non-zero cosmological constant! Either that or the current experiments are dodgy, or GR is wrong!!!

A nice place to end - there are no sacred cows. And in fact we KNOW that GR is only an approximation. We KNOW it doesn't quantise which means that we cannot use it to describe what happens on very small scales when quantum processes are important. So the extreme conditions of the 'singularity' of a black hole and of the initial big bang CANNOT be described by GR. We need quantum gravity......