General Relativity and Cosmology

1 Course Breakdown

1.1 Foundations

1: how to THINK! thought experiment on the roundabout. acceleration curves space(time). Equivalence principle says acceleration = gravity - free fall is an inertial frame. Hence gravity = curved spacetime.
3: tensors - the way to mathematically handle curved space(time)
5: characterising curvature - differentials, geodesic paths are inertial (ie force free) frames and the Riemann curvature tensor
3: gravity caused by energy density - stress energy tensor. so gravity=curved spacetime (the Einstein equations!)

1.2 Experimental tests and black holes

1: Solve the Einstein equations in the special case of a single, spherically symmetric mass - Schwarzchild metric. AND THEN WE SIMPLY USE THIS AND NEVER USE TENSORS AGAIN.
3: Schwarzschild geodesics in weak field (where we can test)
2: Schwarzschild geodesics in strong field - black holes
2/3 of the course uses tensors, 1/3 does not. YOU WILL NEED TO KNOW THEM (but don't give up if you don't - I am always VERY interested in whether you know what it means in physical terms as well as maths). past exams are your guide to my style, but remember the new structure means there is now NO cosmology, and the exam now has 3 long questions, and 8 short ones divided between foundations (2 long, 5 short) and experimental tests (1 long, 3 short)

2 Tensors

These are defined by their transformation properties so a tensor equation is valid in ALL frames. So write physical laws as tensor equations in an inertial frame (special relativity) and then they still work when we change the frame to an accelerating one.

Summation convention $A^\alpha B_\alpha = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3$. Greek letters run from 0-3, mid roman (i,j,k...) from 1-3 (spatial part) and the rest of the alphabet (a,b,c...) from 1-N.

Do transforms between frames with partial derivatives! Definitions of partial derivatives in terms of total derivative

$$\frac{df}{d\tau} = \frac{\partial f}{\partial x}\frac{dx}{d\tau} + \frac{\partial f}{\partial y}\frac{dy}{d\tau} + \frac{\partial f}{\partial z}\frac{dz}{d\tau} = \frac{\partial f}{\partial x^i}\frac{dx^i}{d\tau}$$

REMEMBER HOW TO DO PARTIAL DERIVATIVES! $\partial(1-2m/r)/\partial t \neq d(1-2m/r)/dt$. The first one is ZERO while the second is $2m\dot{r}/r^2$ where $\dot{r} = dr/dt$!

So the frame transformation which relates the new coordinate system $x^{\alpha'}$ to the old one x^β goes as

$$dx^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} dx^\beta$$

Contravariant first order $\quad A^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} A^\beta$

Things like transform like position eg 4-momentum, 4-velocity, 4-force etc.

Covariant first order $\quad A_{\alpha'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} A_\beta$

eg things like the gradient of a scalar field.

second order mixed $\quad A^{\alpha'}_{\beta'} = \frac{\partial x^{\alpha'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\beta'}} A^\mu_\nu$

e.g. $\delta^\mu_\nu = 1$ for $\mu = \nu$ and 0 otherwise. With mixed tensors we can **contract** e.g. $\delta^\nu_\nu = \delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3 = 4$

second order covariant $\quad g_{\alpha'\beta'} = \frac{\partial x^\mu}{\partial x^{\alpha'}}\frac{\partial x^\nu}{\partial x^{\beta'}} g_{\mu\nu}$

eg the metric tensor! This is completely defined by the curvature of space - its how distance relates to position. Its also the way to raise and lower indices (change from covariant to contravariant). $A_\alpha = g_{\alpha\beta} A^\beta$ and $A^\alpha = g^{\alpha\beta} A_\beta$ where $g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha$

3 Curvature

Curved space means that differentials of tensors are tricky - sliding a vector around in flat space means the vector stays parallel to itself. In curved space this isn't true and the final direction of the vector depends on the path taken over the curved space. Condition for parallel transport of some vector - length and direction stay the same along a path s so

$$\frac{d\vec{\lambda}}{ds} = 0 = \frac{d(\lambda^a \vec{e_a})}{ds} = \frac{d\lambda^a}{ds}\vec{e_a} + \lambda^a \Gamma^c_{ab}\frac{dx^b}{ds}\vec{e_c}$$

where $\partial\vec{e_a}/\partial x^b = \Gamma^c_{ab}\vec{e_c}$ - definition of Christoffel symbols. THEY ARE NOT TENSORS as they don't transform as tensors. They show how the basis vectors change over the space and so are related to the metric.

$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}(\partial_a g_{bd} + \partial_b g_{da} - \partial_d g_{ab})$$

so in FLAT space these are zero. Get the derivative onto the same vector to show how components change - this is called **absolute derivative**

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \lambda^b \Gamma^a_b c\frac{dx^c}{ds} = \dot{\lambda}^a + \lambda^b \Gamma^a_{bc}\dot{x}^c$$

where dot denotes derivative wrt s.

Covariant derivative - take the path dependance out: $\lambda^a_{;c} = \partial_c \lambda^a + \lambda^b \Gamma^a_{bc}$
The metric tensor has covariant derivative of zero so we can raise and lower indices in derivative terms as well. $g_{\mu\nu}R^{\nu}{}_{\rho;\sigma} = R_{\mu\rho;\sigma}$

Geodesic paths parallel transport their own tangent vectors. Equivalently they are the shortest distance path between two points. Travelling along a geodesic gives an INERTIAL FRAME - in flat space its a straight line. Parallel transport says components of the tangent vector τ^a go as

$$\frac{d\tau^a}{ds} + \tau^b \Gamma^a_{bc}\frac{dx^c}{ds} = 0$$

but $\tau^a = dx^a/ds$ by definition of a tangent vector so a geodesic path satisfies

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0$$

This holds for any affine parameter u linearly related to s.

NB coordinates can also be related by the metric but remember there is a difference between particles and light. In the weak field we can get gravity out of the geodesic equations!

3.1 Euler-Lagrange equations

In classical mechanics, the Lagrangian

$$L(\dot{x^\mu}, x^\mu) = \frac{1}{2} g_{\mu\nu} \dot{x^\mu} \dot{x^\nu}$$

where dot denotes derivative wrt proper time τ, gives total energy and a geodesic path is the one which minimises this energy. This gives another derivation of the geodesic equations, and gives us the Euler-Lagrange equations on our way.

$$\frac{d}{du}\Big(\frac{\partial L}{\partial \dot{x^\mu}}\Big) - \frac{\partial L}{\partial x^\mu} = 0$$

These can be used either instead of the geodesics equations to get the geodesics OR compare these with the geodesic equations and read off the Christoffel symbols.

3.2 Riemann curvature tensor

Two geodesics separated by distance ζ^a. The rate at which this separation changes determines everything about the curvature of the space.

$$\frac{D^2 \zeta^a}{du^2} + R^a{}_{cbd} \zeta^b \frac{dx^c}{du} \frac{dx^d}{du} = 0 \qquad R^a{}_{cbd} = \Gamma^a_{be} \Gamma^e_{cd} - \partial_d \Gamma^a_{bc} - \Gamma^a_{ed} \Gamma^e_{bc} + \partial_b \Gamma^a_{dc}$$

but Γ^a_{bc} is defined in terms of the derivatives of the metric, so this is all about the 1st and second derivatives of the metric. So its ZERO in flat space. $R^a{}_{cbd}$ also determines the change in orientation of a first order tensor as its parallel transported round a small closed loop.

Contract to form the **Ricci Tensor** $R^a{}_{bca} = R_{bc}$

raise one index and contract again to get **curvature scalar** $R = g^{bc} R_{bc}$

4 Stress-Energy and Einstein Equations

Gravity is caused by energy density. Think of energy density in Special Relativity. Dust where particles have no internal motion - density transforms with frame and energy transforms with frame so needs to be a second order tensor $T^{\nu\mu}$. For perfect fluid in its rest frame then the pressure counts in the energy density as well - POSITIVELY. It adds to gravity (inward force) because it has energy. Conservation of energy and momentum ${T^{\mu\nu}}_{;\nu} = 0$.

Einstein equations: ${R^{\mu\nu}}_{;\nu} = \frac{1}{2}g^{\mu\nu}R$ so to put something about curvature equal to the stress-energy tensor. Full version is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu}$$

where Λ is the cosmological constant (integration constant). An alternative form (setting $\Lambda = 0$) is

$$R^{\mu\nu} = -\frac{8\pi G}{c^4}(T^{\mu\nu} - \frac{1}{2}T)$$

where $T = {T^{\mu}}_{\mu} = g_{\mu\nu}T^{\mu\nu}$

5 Schwarzschild metric

So we get have the Einstein equations. Viciously non-linear. Solve by IMPOSING the form of solution we want. e.g. Schwarzschild metric. Time independent, spherically symmetric EMPTY spacetime around some mass. So $T^{\mu\nu} = T = 0$. And the metric can ONLY depend on r (spherical symmetry) so MUST take the form

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

Get the Christoffel symbols from the Euler Lagrange equations, then stick all this into the Einstein equations to solve for A(r) and B(r). This gives the SCHWARZSCHILD METRIC

$$ds^2 = (1 - 2m/r)c^2dt^2 - (1 - 2m/r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

where $m = GM/c^2$. We use tensors to solve the Einstein equations and get the metric. But then we abandon them!

Proper time is the 'real' time as measured by a clock which travels along the with a particle, proper distance is 'real' distance. COORDINATE TIME AND COORDINATE DISTANCE are not! Coordinates are just a way of linking spacetime in one place with spacetime elsewhere. With $m = 0$ then $g_{\mu\nu} = \eta_{\mu\nu}$ i.e flat Minkowski spacetime. For $m \neq 0$ then at $r = 2m$ the metric goes infinite - Schwarzschild radius.

5.1 Geodesics

Stick to the equatorial plane. Then $d\theta = 0$, $\sin^2\theta = 1$ and the metric simplifies a bit. The Lagrangian gives Euler-Lagrange equations for the geodesics, and/or use the metric for relating coordinates. always simplify the metric according to the physical situation eg radial paths have $d\phi = 0$, circular orbits have $dr = 0$.

radial null geodesics (LIGHT). $d\phi = 0$. This can be done just from the metric. Gives gravitational redshift. The coordinate time difference between signal being emitted E and received R depends only on spatial path. so two signals have the same coordinate time difference. $t_{R1} - t_{E1} = t_{R2} - t_{E2}$ so $t_{E2} - t_{E1} = t_{R2} - t_{R1}$ so $\Delta t(E) = \Delta t(R)$. Then transform coordinate time to proper time and get

$$\frac{\Delta\tau(E)}{\Delta\tau(R)} = \frac{\nu(R)}{\nu(E)} = \Big(\frac{1 - 2m/r(E)}{1 - 2m/r(R)}\Big)^{1/2}$$

(weak field test: Pounds-Rebka gravitational redshift)

Also can use radial null geodesics to calculate the time taken for radial signals to propagate - proper distance is larger AND there is a gravitational time delay. (weak field test: radar signal in solar system)

Elliptical orbits (PARTICLES) - get the equations in terms of $dr/d\phi$ and r. (weak field test: advance of perihelion of mercury)

Lightbending (LIGHT) - again want equations in terms of $dr/d\phi$ and r (weak field test: deflection of light round Sun)

5.2 Geodesics round black holes

Circular orbits (PARTICLES) - get the equations in terms of $d\phi/dt$ and r. Time taken for one orbit is given when $d\phi/dt = 2\pi/t_{orb}$. no circular orbits below r=3m - its where the orbiting particle velocity is the speed of light.
radial null geodesics Infinite redshift at $r = 2m$ - takes infinite proper time for light to get out.
radial particle geodesics (PARTICLES). $d\phi = 0$. get the remaining equations in terms of $dr/d\tau$ and r. This shows that $r = 2m$ is NOT a real singularity as the proper time to fall from $r > 2m$ to $r = 0$ is finite. The real singularity is at $r = 0$. But there are infinite accelerations at $r = 2m$ so to be a stationary obserer is NOT possible here, even with inifinte rocket power.
radial particle and light geodesics for $r < 2m$ Transforming to a frame which moves along with a radially infalling particle shows that light travels inwards even if its emitted radially outwards! not even light can get out so its a black hole.