Foundations and Experimental Tests of GR

1 Course Breakdown

1.1 Foundations

1: how to THINK! thought experiment on the roundabout. acceleration curves space(time). Equivalence principle says acceleration = gravity - free fall is an inertial frame. Hence gravity = curved spacetime.
2-4: tensors - the way to mathematically handle curved space(time)
5-9: characterising curvature - differentials, geodesic paths are inertial (ie force free) frames and the Riemann curvature tensor
10-12: gravity caused by energy density - stress energy tensor. so gravity=curved spacetime (the Einstein equations!)

1.2 Experimental tests and black holes

13: Solve the Einstein equations in the special case of a single, spherically symmetric mass - Schwarzchild metric. AND THEN WE USE THIS AND NOT TENSORS.
14-16: Schwarzschild geodesics - cyclic coordinates and conserved quantities (OK, so a little bit of tensors here!). solution in weak field (where we can test by comparing to Newtonian) - redshift, lightbending, precession of mercury
17: black holes and the nature of the event horizon
2/3 of the course uses tensors, 1/3 does not. YOU WILL NEED TO KNOW THEM (but don't give up if you don't - I am always VERY interested in whether you know what it means in physical terms as well as maths). past exams are your guide to my style, but remember the structure has changed a few times (I started lecturing the course in 2002, cosmology stopped being included after 2003, there have been either 2 or 3 long questions). This years exam has 2 long questions, and 5 short ones divided between foundations and experimental tests.

2 Tensors

These are defined by their transformation properties so a tensor equation is valid in ALL frames. So write physical laws as tensor equations in an inertial frame (special relativity) and then they still work when we change the frame to an accelerating one.
Summation convention $A^\alpha B_\alpha = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3$. Greek letters run from 0-3, mid roman (i,j,k...) from 1-3 (spatial part), rest of the alphabet (a,b,c...) from 1-N.

Do transforms between frames with partial derivatives! Definitions of partial derivatives in terms of total derivative e.g $f = f(x, y, z)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \frac{\partial f}{\partial x^i}\frac{dx^i}{dt}$$

REMEMBER HOW TO DO PARTIAL DERIVATIVES! $\partial(1-2m/r)/\partial t \neq d(1-2m/r)/dt$. The first one is ZERO while the second is $2m\dot{r}/r^2$ where $\dot{r} = dr/dt$!

Position Position vector $\vec{r} = x^\alpha \vec{e}_\alpha$ where $\vec{e}_\alpha$ are basis vectors spanning the space. Can equally be written in another coordinate system: $\vec{r} = x^{\overline{\alpha}} \vec{e}_{\overline{\alpha}}$ where the new coefficiants can be written in terms of the old so $x^{\overline{\alpha}}$ is a function of all the x^β

$$dx^{\overline{\alpha}} = \frac{\partial x^{\overline{\alpha}}}{\partial x^\beta} dx^\beta$$

Contravariant first order $$A^{\overline{\alpha}} = \frac{\partial x^{\overline{\alpha}}}{\partial x^\beta} A^\beta$$

Things that transform like position eg 4-momentum, 4-velocity, 4-force etc.

Covariant first order $$A_{\overline{\alpha}} = \frac{\partial x^\beta}{\partial x^{\overline{\alpha}}} A_\beta$$

eg things like the gradient of a scalar field.

second order mixed $$A^{\overline{\alpha}}_{\overline{\beta}} = \frac{\partial x^{\overline{\alpha}}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\overline{\beta}}} A^\mu_\nu$$

e.g. $\delta^\mu{}_\nu = 1$ for $\mu = \nu$ and 0 otherwise. With mixed tensors we can **contract** e.g. $\delta^\nu{}_\nu = \delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3 = 4$

second order covariant $$g_{\overline{\alpha}\overline{\beta}} = \frac{\partial x^\mu}{\partial x^{\overline{\alpha}}} \frac{\partial x^\nu}{\partial x^{\overline{\beta}}} g_{\mu\nu}$$

most important one is the METRIC tensor. It is formed from the dot product of basis vectors spanning the space $g_{ab} = \vec{e}_a.\vec{e}_b$ This completely defines the curvature of the space - its how distance relates to position, but it also contains information about the coordinate system as well. Its also the way to raise and lower indices (change from covariant to contravariant). $A_\alpha = g_{\alpha\beta} A^\beta$ and $A^\alpha = g^{\alpha\beta} A_\beta$ where $g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha{}^\gamma$

3 Curvature

Curved space means that differentials of tensors are tricky - sliding a vector around in flat space means the vector stays parallel to itself. In curved space this isn't true and the final direction of the vector depends on the path taken over the curved space. Condition for parallel transport is that length and direction stay the same along a path s ie

$$\frac{d\vec{\lambda}}{ds} = 0 = \frac{d(\lambda^a \vec{e_a})}{ds} = \frac{d\lambda^a}{ds} \vec{e_a} + \lambda^a \Gamma^c_{ab} \frac{dx^b}{ds} \vec{e_c}$$

where $\partial \vec{e_a}/\partial x^b = \partial_b \vec{e_a}$ (shorter notation) $= \Gamma^c_{ab} \vec{e_c}$ - definition of Christoffel symbols. THEY ARE NOT TENSORS as they don't transform as tensors. They show how the basis vectors change over the space and so are related to the metric.

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{da} - \partial_d g_{ab})$$

Get the derivative onto the same vector to show how components change - this is called **absolute derivative**

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \lambda^b \Gamma^a{}_{bc} \frac{dx^c}{ds} = \dot{\lambda^a} + \lambda^b \Gamma^a{}_{bc} \dot{x^c}$$

where dot denotes derivative wrt s.

Covariant derivative - take the path dependance out:

$$\frac{D\lambda^a}{ds} = \frac{\partial \lambda^a}{\partial x^c} \frac{dx^c}{ds} + \lambda^b \Gamma^a{}_{bc} \frac{dx^c}{ds} = \lambda^a{}_{;c} \frac{dx^c}{ds}$$

where $\lambda^a{}_{;c} = \partial_c \lambda^a + \lambda^b \Gamma^a{}_{bc}$. Take the partial derivative of the metric tensor

$$\partial_c g_{ab} = \partial_c(\vec{e}_a.\vec{e}_b) = (\partial_c \vec{e}_a).\vec{e}_b + \vec{e}_a.(\partial_c \vec{e}_b) = \Gamma^d_{ac} \vec{e}_d.\vec{e}_b + \vec{e}_a.\Gamma^d_{cb} \vec{e}_d = \Gamma^d_{ac} g_{db} + \Gamma^d_{cb} g_{ad}$$

Thus $\partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{cb} g_{ad} = 0$ i.e. the metric tensor has covariant derivative of zero so we can raise and lower indices in derivative terms as well. $g_{\mu\nu} R^\nu{}_{\rho;\sigma} = R_{\mu\rho;\sigma}$

Geodesic paths parallel transport their own tangent vectors. Equivalently they are the shortest distance path between two points. Travelling along a geodesic gives an INERTIAL FRAME - in flat space its a straight line. Parallel transport says components of the tangent vector τ^a go as

$$\frac{d\tau^a}{ds} + \tau^b \Gamma^a{}_{bc} \frac{dx^c}{ds} = 0$$

but $\tau^a = dx^a/ds$ by definition of a tangent vector so a geodesic path satisfies

$$\frac{d^2 x^a}{ds^2} + \Gamma^a{}_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0$$

This holds for any affine parameter u linearly related to s. These equations can give Newtonian gravity in the weak field, low velocity approximation!

3.1 Euler-Lagrange equations

In classical mechanics, the Lagrangian is $L = T - V$ (KE and gravitational potential). Geodesics are the paths which minimise L. We have no V so $L(\dot{x^\mu}, x^\mu) = \frac{1}{2} g_{\mu\nu} \dot{x^\mu} \dot{x^\nu}$ (where dot denotes derivative wrt time). This formally carries over to GR, and is exactly equivalent to the geodesic equations if dot denotes derivative with respect to any parameter linearly related to proper time (affine parameter). Hence the midstep Euler-Lagrange equations also carry over:

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x^\mu}}\right) - \frac{\partial L}{\partial x^\mu} = 0$$

These are also equivalent to geodesic equations but easier! also can compare these with the geodesic equations and read off the Christoffel symbols - easier than calculating Christoffel symbols from the metric and its derivatives.

3.2 Riemann curvature tensor

Two geodesics separated by distance ζ^a. The rate at which this separation changes determines everything about the curvature of the space. Physically these are TIDAL FORCES

$$\frac{D^2\zeta^a}{du^2} + R^a{}_{cbd}\zeta^b \frac{dx^c}{du}\frac{dx^d}{du} = 0 \qquad R^a{}_{cbd} = \Gamma^a{}_{be}\Gamma^e{}_{cd} - \partial_d\Gamma^a{}_{bc} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} + \partial_b\Gamma^a{}_{dc}$$

but $\Gamma^a{}_{bc}$ is defined in terms of the derivatives of the metric, so this is all about the 1st and second derivatives of the metric. So its ZERO in flat space, irrespective of what coordinate system we used. $R^a{}_{cbd}$ also determines the change in orientation of a first order tensor as its parallel transported round a small closed loop, and determines the difference between order of covariant derivatives $\lambda^a{}_{;bc} - \lambda^a{}_{;cb}$
Contract to form the **Ricci Tensor** $R^a{}_{bca} = R_{bc}$
raise one index and contract again to get **curvature scalar** $R = g^{cb}R_{bc}$ which is NOT the same as $R_{abcd}R^{abcd}$ (Ricci can be zero on CURVED space, as in Schwarzchild solution)

4 Stress-Energy and Einstein Equations

Gravity is caused by energy density. Think of energy density in Special Relativity. Dust where particles have no internal motion (energy is just rest mass) - density transforms with frame and energy transforms with frame so this needs to be a second order tensor $T^{\nu\mu}$. For perfect fluid (rest mass plus pressure i.e. the particles have motion with respect to each other so there is only a global rest frame with no bulk motion, not a frame where every particle is at rest). In this global rest frame then the pressure counts in the energy density as well - POSITIVELY. It adds to gravity (inward force) because it has energy. Conservation of energy and momentum gives $T^{\mu\nu}{}_{;\nu} = 0$.

Einstein equations: $R^{\mu\nu}{}_{;\nu} = \frac{1}{2}g^{\mu\nu}R$ so to get zero derivitive we can write curvature=gravity as

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu}$$

where Λ is the cosmological constant (integration constant). An alternative form (setting $\Lambda = 0$) is

$$R^{\mu\nu} = -\frac{8\pi G}{c^4}(T^{\mu\nu} - \frac{1}{2}T)$$

where $T = T^\mu{}_\mu = g_{\mu\nu}T^{\nu\mu}$

5 Schwarzschild metric

So we get have the Einstein equations. Viciously non-linear. Solve by IMPOSING the form of solution we want. e.g. Schwarzschild metric. Time independent, spherically symmetric EMPTY spacetime around some mass. So $T^{\mu\nu} = T = 0$. And the metric can ONLY depend on r (spherical symmetry) so MUST take the form

$$ds^2 = c^2d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

Get the Christoffel symbols from the Euler Lagrange equations, then stick all this into the Einstein equations to solve for A(r) and B(r). This gives the SCHWARZSCHILD METRIC

$$ds^2 = c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

where $m = GM/c^2$. We use tensors to solve the Einstein equations and get the metric. But then we abandon them!

$d\tau$ is the time as measured by a clock which travels along the worldline with a particle, physical distance is distance along the warped space. COORDINATE TIME AND COORDINATE DISTANCE are simply coordinates! i.e. just a way of linking spacetime in one place with spacetime elsewhere. With $m = 0$ then $g_{\mu\nu} = \eta_{\mu\nu}$ i.e flat Minkowski spacetime. For $m \neq 0$ then at $r = 2m$ then g_{rr} goes infinite and $g_{tt} = 0$ - Schwarzschild radius.

5.1 Geodesics

Particles move in a plane if on geodesics. define this to be equatorial plane without loss of generality as spherical symmetry. Then $d\theta = 0$ and $\sin^2\theta = 1$ and the metric simplifies a bit. The Lagrangian gives Euler-Lagrange equations for the geodesics, and/or use the metric for relating coordinates. Cyclic coordinates (ones which the metric does NOT depend on such as t and ϕ) imply conservation of covariant momentum in that coordinate ($p_t = E$ and $p_\phi = cL_z$ which are related to total energy and angular momentum).

Always simplify the metric according to the physical situation eg radial paths have $d\phi = 0$, circular orbits have $dr = 0$, photon paths are NULL ie $ds^2 = 0$.

radial null geodesics (LIGHT). $d\phi = 0$. This can be done just from the metric. Gives gravitational redshift. The coordinate time difference between signal being emitted E and received R depends only on spatial path. so two signals have the same coordinate time difference. $t_{R1} - t_{E1} = t_{R2} - t_{E2}$ so $t_{E2} - t_{E1} = t_{R2} - t_{R1}$ so $\Delta t(E) = \Delta t(R)$. Then transform coordinate time to proper time and get

$$\frac{\Delta\tau(E)}{\Delta\tau(R)} = \frac{\nu(R)}{\nu(E)} = \Big(\frac{1 - 2m/r(E)}{1 - 2m/r(R)}\Big)^{1/2}$$

(weak field test: Pounds-Rebka-Snyder gravitational redshift)

Also can use radial null geodesics to calculate the time taken for radial signals to propagate - proper distance is larger AND there is a gravitational time delay. (weak field test: radar signals in solar system)

Elliptical orbits (PARTICLES) - use the metric for particles and conservation laws from cyclic coordinates to get the equation for $dr/d\phi$ in terms of r. (weak field test: advance of perihelion of mercury)

Lightbending (LIGHT) - use the metric for photons and conservation laws from cyclic coordinates to get the equation for $dr/d\phi$ in terms of r. (weak field test: deflection of light round Sun)

5.2 Geodesics round black holes

Circular orbits LIGHT - use the simplified metric (no dr and null) and the conservation laws from cyclic coordinates to get $d\phi/dt$ in terms of r. Time taken for one orbit is given when same as above but use null metric. Only one place, at $r = 3m$

Circular orbits PARTICLES -

$d\phi/dt = 2\pi/t_{orb}$. no circular orbits below r=3m. $t^2_{orb} = 4\pi^2 r^3 c^2/m = 4\pi^2 r^3/GM$ is same as newtonian where $v^2/r = (2\pi r/t_{orb})^2 = GM/r^2$

radial null geodesics Infinite redshift at $r = 2m$ - takes infinite proper time for light to get out. event horizon

radial particle geodesics (PARTICLES). $d\phi = 0$. get the remaining equations in terms of $dr/d\tau$ and r. This shows that $r = 2m$ is NOT a real singularity as the proper time to fall from $r > 2m$ to $r = 0$ is finite. The real singularity is at $r = 0$. But there are infinite accelerations at $r = 2m$ so to be a stationary obserer is NOT possible here, even with inifinte rocket power.