1 Experimental tests of GR and Black Holes

The Schwarzschild metric is how gravity curves spacetime IFF the Einstein equations are correct! we saw that they are not derivable from first principles - they are merely the simplest way we can write gravity = curvature. So we need to see if this works, so we need to use this metric to figure out what this spacetime curvature (=gravity) predicts for paths of particles and photons which we can then MEASURE to test the theory. But first, lets look at our metric in a bit more detail.

1.1 Meaning of the Schwarzschild metric

We derived the metric from the Einstein equations, assuming that the cosmological constant is negligible and get

$$ds^2 = c^2 d\tau^2 = c^2(1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

The $dt, dr, d\theta, d\phi$ are COORDINATES, not proper time, or proper distance. What do they mean ?

With $m = 0$ then the metric becomes simply Minkowski in spherical coordinates ie $ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$

Proper time is the time as measured by a clock which travels along with a particle - its obviously invariant as its a property of the particle NOT of the frame! Travelling WITH a particle means that there is no relative spatial motion so $c^2 d\tau^2 = c^2 dt^2$ ie $d\tau = dt$ ie proper time is coordinate time as measured by clocks which are stationary in the reference frame. Now turn m up, and $d\tau^2 = (1 - 2m/r)dt^2$. We are still going to say that clocks record proper time intervals along their world lines, and this clock is fixed, so we have that coordinate time is NOT equal to proper time $d\tau < dt$.

Proper distance is distance of an object measured at teh same time so $dt = 0$. generally we supress the -ve sign in the metric and get proper distance as the spatial part of the metric so $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$. For fixed θ and ϕ then $ds^2 = dr^2$ When we turn up m then the distances go strange - $ds^2 = (1 - 2m/r)^{-1}dr^2$. The proper distance ds IS NO LONGER given by dr, instead we have $ds > dr$. The way to do this is to curve space and can be visualised in an embeding diagram.

Asymptotically, both r and t go to proper distance and proper time.

1.2 Gravitational redshift

We send a light beam along a radial null geodesic. The beam is emitted at r_E and received at r_R. Then the path is

$$0 = c^2(1 - 2m/r)dt^2 - (1 - 2m/r)^{-1}dr^2$$

so it travels at coordinate speed

$$\frac{dr}{dt} = \pm c(1 - 2m/r)$$

so the coordinate time taken (use +ve if going outwards)

$$t_R - t_E = \frac{1}{c}\int_{r_E}^{r_R}(1 - 2m/r)^{-1}dr$$

we could evaluate this but we're not going to as the important this is that it only depends on space. If we send another signal a bit later from the same fixed position emitter to the same fixed position reciever then this coordinate time difference will be the same.

$$t_{R1} - t_{E1} = t_{R2} - t_{E2}$$

so then the coordinate time difference between the two emitted signals an the two received signals is also the same

$$t_{R1} - t_{R2} = t_{E1} - t_{E2} = \Delta t$$

But the clock at the point of emission records proper time not coordinate time. It is fixed in space, so the time it measures between the two signals is

$$\Delta\tau_E = (1 - 2m/r_E)^{1/2}\Delta t_E = (1 - 2m/r_E)^{1/2}\Delta t$$

but so is the clock at the receiver

$$\Delta\tau_R = (1 - 2m/r_R)^{1/2}\Delta t_R = (1 - 2m/r_R)^{1/2}\Delta t$$

so

$$\frac{\Delta\tau_E}{\Delta\tau_R} = \Big(\frac{1 - 2m/r_E}{1 - 2m/r_R}\Big)^{1/2}$$

Suppose the emitter were pulsating, then it has frequency $\nu \propto 1/\Delta\tau$. So

$$\frac{\nu_R}{\nu_E} = \Big(\frac{1 - 2m/r_E}{1 - 2m/r_R}\Big)^{1/2}$$

If $r_E < r_R$ then the photon crawls uphill so it loses energy so $\nu_R < \nu_E$. This can be measured - in the laboratory with a large tower (Pounds–Rebka–Snyder experiment - tiny effect). Or look at absorption lines from the surface of a white dwarf and get a much bigger effect.

1.3 Cyclic coordinates

To find geodesics paths (ie inertial frames!) then the first thing to do is write down the metric - and tailor it to the situation you want to solve. The do the Euler-Lagrange equations. Here we want general paths. These change in both ϕ and r but they are in a PLANE (θ=constant) so without loss of generality as the metric is spherically symmetric we can take this plane to be $\theta = \pi/2$ so the metric is then

$$ds^2 = c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2 d\phi^2$$

Geodesic paths so must satisfy the Euler-Lagrange equation where

$$L = 1/2 g_{\alpha\beta} \dot{x^\alpha} \dot{x}^\beta - (1 - 2m/r)c^2 \dot{t}^2 - (1 - 2m/r)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2$$

The E-L equations are

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$$

where dot denotes derivative with respect to proper time (which is an affine parameter). The easy equations will be those where a coordinate does not appear e.g. t and ϕ as then the Euler Lagrange equations only have one term

$$\frac{d}{d\tau}\Big(\frac{\partial L}{\partial \dot{x}^\alpha}\Big) = 0$$

so $\partial L/\partial \dot{x}^\alpha$ = constant. These are called cyclic coordinates. What is this constant of motion?

$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial(\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)}{\partial \dot{x}^\alpha} = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial \dot{x}^\alpha} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha} \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha}$$

But the metric only depends on position not velocity ie only on x^α not on $\dot{x}^\alpha$ so $\partial g_{\mu\nu}/\partial \dot{x}^\alpha = 0$

$$= \frac{1}{2} g_{\mu\nu} \delta^\mu_\alpha \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \delta^\nu_\alpha = g_{\mu\nu} \dot{x}^\mu = \dot{x}_\nu$$

But momentum is $p^\alpha = \Delta \dot{x}^\alpha$ where Δ is the particle mass. So this constant we are looking for is covariant momentum per unit mass. If coordinate does NOT appear in the metric, then that means that covariant momentum in that coordianate is conserved.

Lets do examples to make it all less abstract. Time coordinate $x^0 = ct$

$$\frac{\partial L}{\partial \dot{x}^0} = (1 - 2m/r)\dot{x}^0 = (1 - 2m/r)\frac{p^0}{\Delta}$$

since this is constant then if we find its value anywhere we know its value everywhere. At $r \to \infty$ this $\to p^0_\infty/\Delta = E_\infty/(\Delta c)$. so

$$(1 - 2m/r)\dot{x}^0 = (1 - 2m/r)c\dot{t} = E_\infty/(\Delta c)$$

$$(1 - 2m/r)\dot{t} = E_\infty/(\Delta c^2) = E$$

where E is a dimensionless measure of the energy - constant zero component of covariant momentum = conservation of something like energy!!!

The other cyclic coordinate (ie one that doesn't appear in the metric) is ϕ. so the E-L equations here become

$$\frac{\partial L}{\partial \dot{x}^3} = \text{constant} = -r^2\dot{\phi}$$

This is indeed covariant angular momentum per unit mass - $p_\phi/Delta = g_{\phi\phi}p^\phi/\Delta = -r^2\dot{\phi}$. More intuitively, this is (modulo a minus sign) like angular momentum $= \Delta r^2 \omega = \Delta r^2 \dot{\phi}$. Thus, angular momentum per unit mass $h = r^2\dot{\phi}$ and $r^2\dot{\phi} = h = cL_z$ ang momentum has units of mass x length x velocity. h is per unit mass so this has units of length x velocity. divide by c and L_z has units of length. But physically this all means something like conservation of angular momentum!!!

1.4 Geodesic Paths - elliptical particle orbits

First thing to do is write down the metric - tailor it to the situation you want to solve. Here we want general paths. These change in both ϕ and r but they are in a PLANE (θ=constant) so without loss of generality as the metric is spherically symmetric we can take this plane to be $\theta = \pi/2$ so the metric is then

$$ds^2 = c^2 d\tau^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2 d\phi^2$$

Geodesic paths so must satisfy the Euler-Lagrange equation where

$$L = 1/2 g_{\alpha\beta} \dot{x^\alpha} \dot{x}^\beta = (1 - 2m/r)c^2 \dot{t}^2 - (1 - 2m/r)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2$$

E-L in t

$$\frac{\partial L}{\partial \dot{t}} = c^2 (1 - 2m/r)\dot{t} = \text{constant}, \quad (1 - 2m/r)\dot{t} = E$$

Similarly for ϕ.

$$\frac{\partial L}{\partial \dot{\phi}} = -r^2 \dot{\phi} = \text{constant}, \quad r^2 \dot{\phi} = cL_z$$

So we have relations for $\dot{\phi}$ and $\dot{t}$. All we need now is one for $\dot{r}$ and we'd have everything we need (there are 3 coordinates so we need 3 equations!). We can get a relation for $\dot{r}$ from the Euler-Lagrance equations but its a bit nasty. All we really need is some way to relate $\dot{r}$ to the known $\dot{\phi}, \dot{t}$, and we can get such a relation more easily through the METRIC which for particles is:

$$c^2 = c^2 (1 - 2m/r)\dot{t}^2 - (1 - 2m/r)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2$$

rearrange to get

$$c^2 (1 - 2m/r)\frac{\dot{t}^2}{\dot{\phi}^2} - (1 - 2m/r)^{-1} \Big(\frac{dr}{d\phi}\Big)^2 - r^2 = \frac{c^2}{\dot{\phi}^2}$$

substitute in from the EL equations for $\dot{t}, \dot{\phi}$,

$$\frac{r^4 E^2}{(1 - 2m/r)L_z^2} - (1 - 2m/r)^{-1} \Big(\frac{dr}{d\phi}\Big)^2 - r^2 = \frac{r^4}{L_z^2}$$

multiply by $-(1-2m/r)$

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L_z^2}(1+L_z^2/r^2)(1-2m/r) - \frac{E^2r^4}{L_z^2} = 0$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{(E^2-1)r^4}{L_z^2} + \frac{2GMr^3}{c^2L_z^2} - r^2 + \frac{2GMr}{c^2}$$

compare this to Newtonian. total energy (KE+PE) per unit mass is $\tilde{E} = \frac{1}{2}v^2 - GM/r$. in spherical polars then $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$ so

$$\tilde{E} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - GM/r$$

again $r^2\dot{\phi} = h$ constant ang. momentum so:

$$\frac{2\tilde{E}}{\dot{\phi}^2} = \left(\frac{dr}{d\phi}\right)^2 + r^2 - \frac{2GM}{r\dot{\phi}^2}$$

$$\frac{2\tilde{E}r^4}{h^2} = \left(\frac{dr}{d\phi}\right)^2 + r^2 - \frac{2GMr^3}{h^2}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{2\tilde{E}r^4}{h^2} - r^2 + \frac{2GMr^3}{h^2}$$

so this is the SAME as $h = cL_z$ (as long as energy means something slightly different $2\tilde{E} = (E^2-1)c^2$ - but we'd kind of expect as newtonian energy doesn't include rest mass and isn't relativistic) EXCEPT for the $2GMr/c^2$ term in the GR equation which is NOT PRESENT in the Newtonian! so orbits in GR should be just a little bit different - this is a TEST of GR, of Einsteins equations which are the simplest way to write gravity=curvature.

1.5 Precession of perihelion of Mercury

need to solve these horrid equations. Easiest to do by putting $u = 1/r$. $du/dr = -1/r^2$. Then $du/d\phi = -1/r^2 dr/d\phi$. The full GR equation is

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{(E^2-1)r^4}{L_z^2} + \frac{2GMr^3}{c^2L_z^2} - r^2 + \frac{2GMr}{c^2}$$

$$\frac{1}{r^4}\Big(\frac{dr}{d\phi}\Big)^2 = \frac{E^2-1}{L_z^2} + \frac{2m}{rL_z^2} - \frac{1}{r^2} + \frac{2m}{r^3}$$

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = \frac{(E^2-1)}{L_z^2} + \frac{2mu}{L_z^2} + 2mu^3$$

Newtonian orbits analogue is

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = \frac{2\tilde{E}}{h^2} + \frac{2GMu}{h^2}$$

where $\tilde{E}$ is a constant related to the energy of the orbit and h^2 is angular momentum per unit mass. $du/d\phi = 0$ at perihelion and aphelion. so Newtonian goes a quadratic

$$u^2 = \frac{2\tilde{E}}{h^2} + \frac{2GMu}{h^2} \quad \text{solve to give} \quad u = \frac{GM}{h^2} \pm \sqrt{\frac{G^2M^2}{h^4} + \frac{2\tilde{E}}{h^2}}$$

so we can see this is elliptical, with mean $u = GM/h^2$. Suppose the orbit is not very elliptical, then setting $y = u - GM/h^2$ means we are looking for SMALL deviations about the mean distance. Substitute in for $u = y + GM/h^2$ and $du/d\phi = dy/d\phi$ so

$$(\frac{dy}{d\phi})^2 + y^2 + \frac{2GMy}{h^2} + \frac{G^2M^2}{h^4} = \frac{2\tilde{E}}{h^2} + \frac{2GMy}{h^2} + \frac{2G^2M^2}{h^4}$$

$$(\frac{dy}{d\phi})^2 + y^2 = \frac{2\tilde{E}}{h^2} + \frac{G^2M^2}{h^4}$$

Trial solution of $y = A\cos(\phi - \phi_0)$, so $dy/d\phi = -Asin(\phi - \phi_0)$ and $(dy/d\phi)^2 = A^2 sin^2(\phi - \phi_0) = A^2 - y^2$ so it works and we could solve for A if we wanted. But the key thing here is that these small deviations from circularity which make the orbit an ellipse are periodic on 2π. The orbit comes back to the same r after 2π so its CLOSED.

Now lets do the SAME THING, but with the GR equation

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = \frac{(E^2-1)}{L_z^2} + \frac{2GMu}{c^2L_z^2} + \frac{2GMu^3}{c^2}$$

set $y = u - GM/h^2$ and replace L_z by remembering $h = cL_z$

$$(\frac{dy}{d\phi})^2 + y^2 = \frac{(E^2-1)}{L_z^2} + \frac{G^2M^2}{h^4} + \frac{2GM}{c^2}(y + GM/h^2)^3$$

$$(\frac{dy}{d\phi})^2 + y^2 = \tilde{E} + \frac{G^2M^2}{h^4} + \frac{2GM}{c^2}(y^3 + \frac{3GM}{h^2}y^2 + \frac{3G^2M^2}{h^4}y + \frac{G^3M^3}{h^6})$$

neglect terms in y^3 - these are SMALL deviations.

$$(\frac{dy}{d\phi})^2 + (1 - \frac{6G^2M^2}{h^2c^2})y^2 = \tilde{E} + \frac{G^2M^2}{h^4} + \frac{2G^4M^4}{h^6c^2} + \frac{6G^3M^3y}{h^4c^2}$$

The trial solution here is $y = y_0 + A\cos(k\phi - \phi_0)$ so $dy/d\phi = -Ak\sin(k\phi - \phi_0)$ and

$$\left(\frac{dy}{d\phi}\right)^2 = A^2k^2\sin^2(k\phi - \phi_0) = A^2k^2(1 - \cos^2(k\phi - \phi_0) = A^2k^2 - k^2(y - y_0)^2$$

$$= A^2k^2 - k^2y^2 + 2k^2y_0y - k^2y_0^2$$

This is going to work as we now have a multiplier for y^2 and y as well as a constant term. Equating coefficients for y^2 gives

$$-k^2 + (1 - \frac{6G^2M^2}{h^2c^2}) = 0$$

$$k = (1 - \frac{6G^2M^2}{h^2c^2})^{1/2}$$

So we get round to the same r NOT when $\phi = 2\pi$ but when $k\phi = 2\pi$ ie at

$$\phi = \frac{2\pi}{k} = 2\pi(1 - \frac{6G^2M^2}{h^2c^2})^{-1/2} \approx 2\pi(1 + \frac{3G^2M^2}{h^2c^2})$$

from a binomial expansion. Then the change in ϕ in one orbit is

$$\Delta\phi = \phi - 2\pi = 6\pi G^2M^2/(h^2c^2)$$

So orbits in GR are not (quite) closed. An elliptical orbit will have the closest point precess around. So first we need this consstant $h = r^2\dot{\phi} = r^2 2\pi/P$ where P is the period. In Newtonian this is $P^2 = 2\pi r^3/(GM)$ so $h^2 = r^4 GM/r^3 = GMr$ so $\Delta\phi = 6\pi G^2M^2/(GMrc^2) = 6\pi GM/(rc^2)$. For Mercury (closest planet so one in the strongest gravitational field, ie where this effect will be biggest) then $r \sim 5.2 \times 10^{10}m$ so $\Delta\phi = 5 \times 10^{-7}$ radians per orbit. This is TINY!! but its also cumulative. One orbits precession is difficult to measure, but after 100 years ie 415 orbits then the point of closest approach (perihelion) has shifted by 0.01189 radians or 43 arcseconds. This is measurable, and was a well known problem in the 19th century that newtonian mechanics didn't get the right answer for Mercury's orbit. GR does!!!

1.6 Geodesic paths - Light

First thing to do is write down the metric - tailor it to the situation you want to solve. Here we want general paths. These change in both ϕ and r but they are for PHOTONS so the spacetime interval is NULL

$$ds^2 = c^2 d\tau^2 = 0 = (1-2m/r)c^2 dt^2 - (1-2m/r)^{-1} dr^2 - r^2 d\phi^2$$

so we just have to be cunning about what we choose as an affine parameter - its NOT $d\tau$, but there is some affine parameter we can use, so we can still write a Lagrangian and do the E-L equations for the cyclic coordinates.

$$L = 1/2 g_{\alpha\beta} \dot{x^\alpha} \dot{x}^\beta = (1-2m/r)c^2 \dot{t}^2 - (1-2m/r)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2$$

E-L in t

$$\frac{\partial L}{\partial \dot{t}} = c^2(1-2m/r)\dot{t} = \text{constant}, \quad (1-2m/r)\dot{t} = E$$

Similarly for ϕ.

$$\frac{\partial L}{\partial \dot{\phi}} = -r^2\dot{\phi} = \text{constant}, \quad r^2\dot{\phi} = cL_z$$

Again, we could do the r E-L equation to get a 3rd expression for our 3 unknown coordinates, but this would lead to $\ddot{r}$ whereas if we use the metric to relate them we only have to deal with $\dot{r}$.

$$0 = c^2(1-2m/r)\dot{t}^2 - (1-2m/r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$

rearrange to get

$$c^2(1-2m/r)\frac{\dot{t}^2}{\dot{\phi}^2} - (1-2m/r)^{-1}\Big(\frac{dr}{d\phi}\Big)^2 - r^2 = 0$$

substitute in from the EL equations for $\dot{t}, \dot{\phi}$,

$$\frac{r^4 E^2}{(1-2m/r)L_z^2} - (1-2m/r)^{-1}\Big(\frac{dr}{d\phi}\Big)^2 - r^2 = 0$$

multiply by $-(1-2m/r)$

$$\Big(\frac{dr}{d\phi}\Big)^2 + r^2(1-2m/r) - \frac{E^2 r^4}{L_z^2} = 0$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{E^2 r^4}{L_z^2} - r^2 + \frac{2GMr}{c^2}$$

Again its easier (not easy!) to solve in $u = 1/r$ so $du = -1/r^2 dr$ and

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2}{L_z^2} - u^2 + \frac{2GMu^3}{c^2}$$

when $GM/R = GMu \to 0$ i.e. weak gravity, this gives ($F = E^2/L_z^2$)

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{E^2}{L_z^2}$$

which has a particular solution of $u = u_0 \sin\phi$ where $u_0 = E/L_z$. Put this in terms of r and get $r\sin\phi = 1/u_0 = r_0 = L_z/E$. This is just a straight line with impact parameter - $r_0 = b = L_z/E$ - perpendicular distance far from the object. so a straight line is what Newtonian gravity predicts about light (if light is not affected by gravity)! Actually its a bit ambiguous in Newtonian gravity - light is massless so not affected, but gravitational force is INDEPENDANT of mass so should be affected....

Now lets do the weak field approximation again

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{E^2}{L_z^2} + \epsilon u^3$$

at closest approach then $u = u_0$ again and $du/d\phi = 0$ so

$$u_0^2 = E^2/L_z^2 + \epsilon u_0^3$$

we can use this to substitute for E^2/L_Z^2 in the general equation

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = u_0^2(1 - \epsilon u_0) + \epsilon u^3$$

This should have a solution which is just a small perturbation of the flat spacetime solution - maybe $u = u_0 \sin\phi + \epsilon f(\phi)$ where f is some function of ϕ to be determined. Stick this in, and ignore anything with higher powers of ϵ

$$2\cos\phi\frac{df}{d\phi} + 2f\sin\phi = u_0^2(\sin^3\phi - 1)$$

This is just a first order differential equation of f, so we'll go look up the solution.

$$f = \frac{1}{2}u_0^2(1 + \cos^2\phi - \sin\phi) + A\cos\phi$$

A is a constant of integration so can be found from the boundary conditions. At $r = \infty$ then $\phi = 0$ and $f = 0$, so $A = -u_0^2$. So our full solution for the weak field effect of gravity on the path of a light beam is

$$u = u_0(1 - \frac{1}{2}\epsilon u_0)\sin\phi + \frac{1}{2}\epsilon u_0^2(1 - \cos\phi)^2$$

So our full solution for the weak field effect of gravity on the path of a light beam is

$$u = u_0(1 - \frac{1}{2}\epsilon u_0)\sin\phi + \frac{1}{2}\epsilon u_0^2(1 - \cos\phi)^2$$

so we no longer expect a straight line as the photon goes from $\phi = 0$ to π. As $r \to \infty$ i.e. $u \to 0$ then $\phi \to \pi + \alpha$ where $\alpha \ll 1$. $\cos(\pi + \alpha) = \cos\pi\cos\alpha - \sin\pi\sin\alpha \approx -1$ while $\sin(\pi + \alpha) = \sin\pi\cos\alpha + \cos\pi\sin\alpha = -\sin\alpha \approx -\alpha$.

$$0 = -(1 - \frac{1}{2}\epsilon u_0)\alpha + 2\epsilon u_0 \approx -\alpha + 2\epsilon u_0$$

$$\alpha \sim 2\epsilon u_0 = 4\frac{GM}{r_0 c^2}$$

Take r_0 as equal to the suns radius and look at a star in the line of sight just away from the limb of the sun and then the prediction for the deflection of its position is 1.75 seconds of arc. This was the first test of GR, measuring positions of stars close to the sun as seen during a solar eclipse and comparing these to their positions as seen when the sun was nowhere near the line of sight.

Incidentally, if Newtonian gravity DOES affect light, with the mass independant acceleration term of GM/r^2, then the prediction for lightbending is exactly half of the GR one (AND it would predict that light would accelerate i.e. go faster than c!!). The observations come out on the GR prediction and can rule out EITHER of the Newtonian predictions.

2 Black Holes

We've seen that GR works in the wimpy weak field approximations. so lets see what it predicts in strong fields..... in particular something funny happens in the metric at $r = 2m$. the coordinate time interval $\to 0$ and the radial interval $\to \infty$. is this a true singularity or a coordinate one? eg in polar coordinates, using θ and ϕ on the surface of a sphere, then there is a coordinaet singularity at $\theta = 0, \pi$ i.e. at the poles. This is a single point for all values of ϕ so the coordinate mapping is not one-to-one. yet nothing PHYSICAL happens here. I could choose the origin for my coordinate system at any point on the surface of a sphere. so is the singularlity in the metric at $r = 2m$ 'real' or is it just a coordinate singularity?

2.1 Nature of the Event Horizon: radial geodesics

Go into a black hole on a radial free fall geodesic. Radial geodesic with $d\theta = d\phi = 0$

$$c^2 = c^2(1 - 2m/r)\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2$$

Substitute in from the particle EL equation for t, $(1 - 2m/r)\dot{t} = E = E_\infty/(mc^2)$ and we get

$$\dot{r}^2 - c^2E^2 + c^2(1 - 2m/r) = 0$$

so the velocity depends on E, i.e. depends on whether it starts from rest and drops towards the hole, or whether its flung with some velocity towards the hole.

For a particle dropped at rest from $r >> 2m$ then $dt \approx d\tau$ so $E^2 = (1 - 2m/r)\dot{t} \approx 1$ Substitute this back into our free fall geodesic and we have

$$\dot{r}^2 - c^2 + c^2(1 - 2m/r) = 0$$

$$\dot{r} = \pm c(2m/r)^{1/2}$$

We have to take a square root so we can have either sign for $\dot{r}$. Since particles fall down then it makes most sense to take the -ve one so

$$\dot{r} = \frac{dr}{d\tau} = -\Big(\frac{2mc^2}{r}\Big)^{1/2}$$

It looks the same as the standard Newtonian expression but here we are taking the derivative with respect to proper time τ NOT coordinate time t, and the radial coordinate is NOT the same as radial distance.

So we can now calculate how long it would take to fall in the gravitational field from r_1 to r_2. $\tau = \int_{r_1}^{r_2} d\tau/dr dr$ so

$$\tau = \frac{1}{c\sqrt{2m}} \int_{r_2}^{r_1} r^{1/2} dr = \frac{2}{3c\sqrt{2m}} (r_1^{3/2} - r_2^{3/2})$$

THIS HAS NO SINGULARITY anywhere. so I can have $r_1 = 10m$ and $r_2 = 2m$ and find it takes proper time $\sim 13.6m/c$ (time as measured by a clock on this worldline) to fall to the horizon. It is smoothly varying across $r = 2m$ and we could have done this with $r_1 = 10m$ and $r_2 = 0$ and get proper time of $14.9m/c$! so it doesn't take too long to fall down to the singularity and be crushed out of existance. It is smoothly varying across $r = 2m$ and we can do it again, taking $r_1 = 2m$ and r_2 down to zero! and get proper time of $4m/3c$! so it doesn't take too long to fall down to the singularity!

But coordinate time is nothing like so nice. The Euler lagrange equation relates proper time and coordindate time for the particle, and this gives $(1 - 2m/r)\dot{t} = E = 1$ for a particle starting from rest at infinity so $dt/d\tau = 1/(1 - 2m/r)$ and the coordinate speed is

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = -c(\frac{2m}{r})^{1/2}(1 - 2m/r)$$

This drops to zero at $r = 2m$! So the coordinate time interval goes infinite as $r \to 2m$. So it takes an infinite amount of coordinate time to fall to $r = 2m$!! We saw from the redshift argument that coordinate time intervals between 2 photons emitted at 1 radius are the same as the coordinate time intervals between 2 photons received at another radius, and also that at $r \to \infty$ then $dt \to d\tau$ so to an observer sitting a long way away the person falling into the black hole gets slower and slower as they approach the event horizon. They appear to hover on the horizon for ever, while any light/signals emitted from them gets infinitely redshifted as $r_2 \to 2m$. The redshift (ie observed frequency) goes imaginary for $r_2 < 2m$. But $E = h\nu$ so no light/energy can get out from within $r = 2m$.

2.2 Nature of event horizon: acceleration

What is happening at $r = 2m$?? we can get some insight by looking at the acceleration. Dropping from rest at $r = r_0$ - substitute back in the metric to find the new value of E for this case.

$$(1 - 2m/r)c^2\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2 = c^2$$

$$(1 - 2m/r_0)c^2E^2/(1 - 2m/r_0)^2 = c^2$$

so $E^2 = (1 - 2m/r_0)$. Put this back in the metric to find the COORDINATE speed.

$$(1 - 2m/r)c^2 - (1 - 2m/r)^{-1}\Big(\frac{dr}{dt}\Big)^2 = c^2\frac{(1 - 2m/r)^2}{(1 - 2m/r_0)}$$

$$\Big(\frac{dr}{dt}\Big)^2 = c^2(1 - 2m/r)^2\Big(1 - \frac{1 - 2m/r}{1 - 2m/r_0}\Big)$$

$$= c^2(1 - 2m/r)^2\frac{2m/r - 2m/r_0}{1 - 2m/r_0}$$

$$\frac{dr}{dt} = -c(1 - 2m/r)\Big(\frac{2m/r - 2m/r_0}{1 - 2m/r_0}\Big)^{1/2}$$

As this falls past a STATIONARY observer at r then we know how to relate proper distance, dR_s, and proper time $d\tau_s$ to coordinate time intervals by the metric $dR_s = (1 - 2m/r)^{-1/2}dr$ and $d\tau_s = (1 - 2m/r)^{1/2}dt$. so this observer measures a proper velocity

$$\frac{dR_s}{d\tau_s} = \frac{dR_s}{dr}\frac{dr}{dt}\frac{dt}{d\tau_s} = (1-2m/r)^{-1}\frac{dr}{dt} = -c(1-2m/r)\Big(\frac{2m/r - 2m/r_0}{1 - 2m/r_0}\Big)^{1/2}(1-2m/r)^{-1}$$

$$= -c\Big(\frac{2m/r - 2m/r_0}{1 - 2m/r_0}\Big)^{1/2}$$

so if the observer is at $r = 2m$ they see the particle falling at the speed of light, irrespective of the r_0 from which it was dropped. So even if $r_0 = 2m + \epsilon$ where $\epsilon << 1$ then by $r = 2m$ the particle is traveling at c. since it goes from rest to c then the acceleration goes infinite at $r = 2m$. there are infinite accelerations measured by a stationary observer - which just goes to show that the point is that there ARE NO stationary observers possible at $r = 2m$. you cannot stay still at the horizon, and watch things come past, you HAVE to move downwards.

2.3 Nature of the The Event Horizon

Lets pull it all together. At $r = 2m$ there is infinite acceleration, so no such thing as fixed anything. So its NOT just a coordinate singularity, physically there is something going on here, and this is supported by an observer at infinity watching an infalling particle - they see it initially accelerate towards the hole but as it gets closer then it slows down and appears to STOP at the horizon. But its not a true singularity - on an infalling particle it takes a finite time to get to $r = 0$ rather than running into any real barrier at $r = 2m$.

The way to think about this is to look at the scalar we can form from the Riemman curvature tensor $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$. This is NOT the same as the scalar we formed from contracting the Ricci tensor $R^{\mu}{}_{mu} = g_{\mu\nu}R^{\mu\nu}$ - We saw that for empty space outside a gravitating body then this (and the Ricci tensor itself) were ZERO as the stress energy tensor and scalar are zero if space is EMPTY and the Einstein equations can be written $R^{\mu\nu} = \kappa(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T)$. But the ricci tensor is a contraction of the full Riemann curvature and it is NOT necessarily true that all components of the Riemann curvature are zero if $R^{\mu\nu} = 0$ - there is real curvature of spacetime (there has to be as this is GRAVITY). This Riemann curvature scalar $\propto r^{-6}$. There IS a true curvature singularity, and its at $r = 0$ as expected.

If we draw the curvature of spacetime (rather than an embedding diagram which shows proper distance versus r) then we can see whats going on. Any freefalling observer is in an inertial frame - we can draw a flat bit of spacetime which is tangent to the spacetime curvature at that point. and since time is orthoganal to space then we can use this to define the t axis. and lightcones make a 45 degree angle as normal - its all locally intertial and fine and special relativity works, but its TILTED with respect to ∞ due to the spacetime curvature. at $r = 2m$ the tilt is 45 degrees, so light DOES NOT GET OUT. There is no way that lightcones can get out to infinity - they simply don't connect. so a black hole is simply spacetime so curved that not even light can get out.

2.4 Particle orbits in strong gravity

This is also an exam question from 2005.

$$ds^2 = c^2 d\tau^2 = (1-2m/r)c^2dt^2 - (1-2m/r)^{-1}dr^2 - r^2 d\phi^2$$

cyclic coordinates $(1-2m/r)\dot{t} = E$ and $r^2\dot{\phi} = cL_z$ substitute back into it teh metric and solve for $\dot{r}$

$$c^2 = (1-2m/r)c^2\dot{t}^2 - (1-2m/r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$

$$c^2(1-2m/r) = c^2E^2 - \dot{r}^2 - c^2L_z^2(1-2m/r)/r^2$$

$$\dot{r}^2 = c^2E^2 - c^2(1+L_z^2/r^2)(1-2m/r)$$

$$\dot{r}^2 = c^2E^2 - c^2V^2(r)$$

where $V^2(r) = (1+L_z^2/r^2)(1-2m/r)$. this is the effective potential - the terms represent an angular momentum barrier and gravity.

We can get more insight into this by contrasting with Newtonian - angular momentum barrier and gravity give an effective potential of

$$V(r) = \frac{h^2}{2r^2} - GM/r$$

The first term is the angular momentum barrier - particle orbits around a large mass - it wants to get closer because gravity is attractive, but as it gets closer then by conservation of angular momentum it goes faster so there is a bigger force outwards. This $\to \infty$ as $r \to 0$, and $\to 0$ as $r \to \infty$

In GR then the potential is a bit diffferent to the Newtonian shape. V^2 - zero at $r = 2m$, $\to -\infty$ as $r \to 0$ and $\to 1$ as $r \to \infty$. turning points where $dV^2/dr = 0$.

$$(1+L_z^2/r^2) - 2m - 1/r^2 + -2L_z^2/r^3(1-2m/r) = 0$$

$$(1+L_z^2/r^2)2m/r^2 = 2L_z/r^3(1-2m/r)$$

$$mr^2 + L_z^2 = L_z^2 r(1-2m/r)$$

$$mr^2 - L_z^2 r + 3mL_z^2 = 0$$

$$r = L_z^2/2m \pm L_z/2m\sqrt{L_z^2 - 12m^2}$$

So in general has 2 turning points. Do d^2V^2/dr^2 and see that the maximum is at smaller r, minimum at larger r.

For the specific case of $L_z = 4m$ then the turning points are at $8m \pm 4m$ i.e $r = 4m$ and $12m$. At $r = 4m$ $V^2 = 1$, while at $r = 12m$ $V^2 = 35/36$. Hence a particle with $L_z = 4m$ and $E^2 > 1$ can get over all the potential barriers so it can get everywhere from $r = 0$ to ∞. So the path of this particle ends up down the black hole.

For $L_z = \sqrt{12}m$ then the two turning points merge together, with both at $r = 6m$. $d^2V^2/dr^2 = 0$ so this is neither a maximum nor a minimum - its a point of inflection. Stable orbits require a MINIMUM in the potential and that the particle energy E^2 is less than V^2 on either side of the minimum so that the particle is confined to a range of radii (elliptical orbit). If instead the particle energy is exactly at the minimum of the potential then there is no range in radii accessible to the particle and its a circular orbit. If E^2 is such that there is a potential barrier at small r, but NOT at larger r then this is an unbound hyperbolic orbit.

Anyway, for a point of inflection there is no minimum. so this is NOT a stable orbit. Its unstable - any small perturbation and the particle will spiral in towards the black hole. $r = 6m$ is the last stable orbit around a Schwarzschild black hole. It is NOT where a particle would orbit around at the speed of light - thats at $r = 3m$. There are orbits that are possible between $r - 6m$ and 3m but THEY ARE NOT STABLE. Gravity is just stronger in GR than in Newtonian - the centrifugal barrier is simply not big enough to hold up any more.