Last lecture we that not all physical quantities transform like the coordinates ie as contravariant first order tensors (4-vectors). Gradients go as the inverse transform of the coordinates - these are covariant first order tensors. And then we had the *metric tensor*, the way to get distance between any two points, which transforms as TWO inverse transform of the coordinates - its a second order covariant tensor. We started to use this to describe a CURVED space - the surface of a sphere. We found the basis vectors for our curved surface, and they make sense in terms of what we expect to see. Now lets make the metric tensor components:

2.9 Example metric for the surface of a sphere(continued)

The metric tensor components are the dot products of the basis vectors so

$$g_{\overline{11}} = \vec{e_{\overline{1}}}.\vec{e_{\overline{1}}} = a^2\cos^2 x^{\overline{1}}\cos^2 x^{\overline{2}} + a^2\cos^2 x^{\overline{1}}\sin^2 x^{\overline{2}} + a^2\sin^2 x^{\overline{1}} = a^2$$

$$g_{\overline{12}} = \vec{e_{\overline{1}}}.\vec{e_{\overline{2}}} = -a^2\cos x^{\overline{1}}\cos x^{\overline{2}}\sin x^{\overline{1}}\sin x^{\overline{2}} + a^2\cos x^{\overline{1}}\sin x^{\overline{2}}\sin x^{\overline{1}}\cos x^{\overline{2}} = 0$$

$$g_{\overline{22}} = \vec{e_{\overline{2}}}.\vec{e_{\overline{2}}} = a^2\sin^2 x^{\overline{1}}\sin^2 x^{\overline{2}} + a^2\sin^2 x^{\overline{1}}\cos^2 x^{\overline{2}} = a^2\sin^2 x^{\overline{1}}$$

So the line element showing us how distance relates to changing coordinates on the surface of a sphere is the familiar one

$$ds^2 = g_{\overline{AB}}dx^{\overline{A}}dx^{\overline{B}} = a^2(dx^{\overline{1}})^2 + a^2\sin^2 x^{\overline{1}}(dx^{\overline{2}})^2 = a^2d\theta^2 + a^2\sin^2\theta d\phi^2$$

Well, its familar if you can remember anything about spherical polar coordinates!

Here we have used our knowledge of the curvature of space to define the metric. Alternatively we can of course go the other way and use the metric to DEFINE how distance between points relates to coordinates. We've already seen one metric tensor which embodies flat spacetime, that of special relativity where $g_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$. Its is NOT positive definite. $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. In mathematical terms, a manifold is some N dimensional space which can be characterised by a metric tensor (ie coordinates can be related to distance and its differentiable). Reimanian manifolds have positive definite distance, pseudo-Reimanian do not. So the familar spacetime can be terms 'a 4D pseudo-Reimanian manifold'. I'll just stick to spacetime.

2.10 Basis vectors for covariant components

We got the basis vectors for the contravariant components by getting the vectors tangent to the coordinate curves. We've been working with these basis vectors which define a tangent space T_P at P. A contravariant vector $\vec{\lambda} = \lambda^a \vec{e_a}$ is an arrow emmanating from P - its not in the manifold like the curve is in the manifold, but is attached to it like the tangent space is attached to a surface.

But we could instead have defined a set of basis vectors by the vectors which were normal to the coordinate surfaces rather than tangent to the coordinate curves. We get a coordinate surface by holding that coordinate constant. and then we want to get the normal to this surface. We get normals by taking the GRADIENT. We can regard each coordinate as a scalar field Then the vector perpendicular to this scalar field is $\vec{e^a} = \nabla x^a$. And if we do a transformation of coordinates $x^{\overline{b}} = x^{\overline{b}}(x^a)$ then these basis components transform as

$$\vec{e^{\overline{b}}} = \nabla x^{\overline{b}} = \frac{\partial x^{\overline{b}}}{\partial x^a} \nabla x^a = \frac{\partial x^{\overline{b}}}{\partial x^a} \vec{e^a}$$

i.e. these transform in the opposite way to our tangent basis vectors.

This gives us another vector space of the manifold - call it T_P^* the dual or cotangent space of the manifold at P. The basis vectors themselves look identical if we have an orthonormal set of coordinates (they don't have to be reclilinear, just be 90 degrees where they meet). BUT THEY ARE NOT IDENTICAL IF THE COORDINATES ARE NOT ORTHOGANAL.

2.11 Covariant and contravariant: more on the metric

But if we have another set of basis vectors then we can write any arbitrary vector either on the old basis in the tangent space OR the new basis in the cotangent space i.e. $\vec{\lambda} = \lambda^a \vec{e_a} = \lambda_b \vec{e^b}$. If the basis vectors are the same i.e. we had orthonormal bases then the contravariant and covariant components are IDENTICAL. But of course in general they are not.

For the pure mathematically inclined, setting the contravariant and covariant forms of writing this as equal is a bit of a fudge as $\lambda^a \vec{e_a}$ and $\lambda_b \vec{e^b}$ are actually are in two different vector spaces T_P and $T*_P$, respectively. But we'll do a bit more on this later

Take another vector $\vec{\mu} = \mu^a \vec{e_a} = \mu_a \vec{e^a}$. The dot product of these is

$$\vec{\lambda}.\vec{\mu} = \lambda^a \vec{e_a}.\mu^b \vec{e_b} = \lambda^a \mu^b \vec{e_a}.\vec{e_b} = \lambda^a \mu^b g_{ab}$$

But we could have done this with the covariant components alone

$$\vec{\lambda}.\vec{\mu} = \lambda_a \vec{e^a}.\mu_b \vec{e^b} = \lambda_a \mu_b \vec{e^a}.\vec{e^b} = \lambda_a \mu_b g^{ab}$$

where we have defined $g^{ab} = \vec{e^a}.\vec{e^b}$. Again this is symmetric by definition of the dot product so $g^{ab} = g^{ba}$. But we could also have done this with the mixed components

$$\vec{\lambda}.\vec{\mu} = \lambda^a \vec{e_a}.\mu_b \vec{e^b} = \lambda^a \mu_b \vec{e_a}.\vec{e^b}$$

What is this dot product ? lets do it explicitally in 3D

$$\vec{e_i}.\vec{e^j} = \frac{\partial \vec{r}}{\partial x^i}.\nabla x^j$$

where $\vec{r} = x^i \vec{e_i}$. Then we have

$$\begin{aligned}
e_i.e^j &= (\frac{\partial x^1}{\partial x^i} i + \frac{\partial x^2}{\partial x^i} j + \frac{\partial x^3}{\partial x^i} k).(\frac{\partial x^j}{\partial x^1} i + \frac{\partial x^j}{\partial x^2} j + \frac{\partial x^j}{\partial x^3} k) \\
&= \frac{\partial x^1}{\partial x^i}\frac{\partial x^j}{\partial x^1} + \frac{\partial x^2}{\partial x^i}\frac{\partial x^j}{\partial x^2} + \frac{\partial x^3}{\partial x^i}\frac{\partial x^j}{\partial x^3} \\
&= \frac{\partial x^j}{\partial x^i} = \delta_i^j
\end{aligned}$$

where the kronekar delta function is $\delta_i^j = 1$ for $i = j$, 0 otherwise. NB because of the summation convention then $\delta_i^i = \delta_1^1 + \delta_2^2 + \delta_3^3 = 3$ So now we have

$$\lambda^a \vec{e_a}.\mu_b \vec{e^b} = \lambda^a \mu_b \vec{e_a}.\vec{e^b} = \lambda^a \mu_b \delta_a^b = \lambda^a \mu_a$$

By symmetry of the dot product and metric then $\lambda^a \mu_a = \lambda_a \mu^a$. So collecting all these together we have

$$\lambda^a \mu_a = \lambda_a \mu^a = g_{ab} \lambda^a \mu^b = g^{ab} \lambda_a \mu_b$$

So this gives us an easy way of swapping between contravariant and covariant components - we use the METRIC $g_{ab}\lambda^a \mu^b = \lambda_a \mu^a$ so

$$\lambda_a = g_{ab} \lambda^b$$

and as $g^{ab}\lambda_a\mu_b = \lambda^b\mu_b$ so

$$\lambda^a = g^{ab}\lambda_b$$

Tensors derived from other tensors by raising or lowering the indices via the metric are called **associated tensors**.

We also get another useful metric relation in that $\lambda_a = g_{ab}\lambda^b = g_{ab}g^{bc}\lambda_c$ so

$$g_{ab}g^{bc} = \delta^a_c$$

For the pure mathematically inclined, heres how its done. A contravariant vector $\vec{\lambda} = \lambda^a\vec{e_a}$ in T_p is associated with a covariant vector $\tilde{\lambda} = \lambda_a\vec{e^a}$ in $T*_P$ via the metric (or dot or inner product) such that for all vectors $\vec{\mu}$ in T_P then $\tilde{\lambda}(\vec{\mu})$ is a real number $= \mu^a\lambda_a$. i.e. it define covariant vectors as linear functions which map covariant vectors (one–forms) into real numbers via the metric. Schutz takes this approach (eventualy), as do Misner, Thorne and Wheeler. Foster and Nightingale take the approach I've used here.